

The Exact S -Matrix for an $osp(2|2)$ Disordered System

Zorawar S. Bassi*
André LeClair

Newman Laboratory
Cornell University
Ithaca, NY 14853, USA

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Abstract

We study a two-dimensional disordered system consisting of Dirac fermions coupled to a scalar potential. This model is closely related to a more general disordered system that has been introduced in conjunction with the integer quantum Hall transition. After disorder averaging, the interaction can be written as a marginal $osp(2|2)$ current-current perturbation. The $osp(2|2)$ current-current model in turn can be viewed as the fully renormalized version of an $osp(2|2)^{(1)}$ Toda-type system (at the marginal point). We build non-local charges for the Toda system satisfying the $U_q[osp(2|2)^{(1)}]$ quantum superalgebra. The corresponding quantum group symmetry is used to construct a Toda S -matrix for the vector representation. We argue that in the marginal (or rational) limit, this S -matrix gives the exact (Yangian symmetric) physical S -matrix for the fundamental “solitons” of the $osp(2|2)$ current-current model.

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*zorawar@mail.lns.cornell.edu

1. Introduction

Non-perturbative techniques have proven to be very useful in the investigation of interacting field theories. This is especially true in two-dimensions, where rich symmetry structures have allowed the calculation of fundamental quantities, such as exact S -matrices and correlation functions, for many important models. In recent times, this progress has been further advanced by a better understanding of the mathematical framework underlying the symmetry structures. Perhaps this is most apparent in the construction of exact S -matrices for integrable 2D systems. Here a knowledge of the symmetry algebra and its representation theory is essential to building a S -matrix satisfying the Yang-Baxter constraint [1–6]. Well known examples of models with exact S -matrices include the sine-Gordon/massive Thirring model [7–9], Gross-Neveu models [10–12], principal chiral and sigma-type models [13–16], various statistical systems, such as RSOS systems [17,18], and Toda theories based on Lie algebras [19–24]. A relatively new class of models that have been investigated are based on Lie superalgebras. Supersymmetric Toda models belong to this category and their exact S -matrices have been calculated [25,26].

In this paper we use the S -matrix approach to study a model based on the Lie superalgebra $osp(2|2)$, which is closely related to a disordered system introduced to describe the integer quantum Hall transition [27]. As shown by Bernard [28], the model arises after disorder averaging over a random scalar potential and consists of a free fermionic and bosonic piece, combined with a marginal $osp(2|2)$ current-current perturbation. The bosonic part is the result of rewriting the fermion partition function as a path integral over complex bosonic variables. This pairs the fermions and bosons thus making the action “supersymmetric”. The model is integrable with factorized scattering and Yangian symmetry. By introducing an anisotropy, which allows us to flow between a relevant and marginal perturbation, we construct non-local charges for half the current-current operators. The remaining current-current operators are generated under renormalization at the marginal point. The non-local charges are shown to satisfy the $U_q[osp(2|2)^{(1)}]$ quantum superalgebra and to be conserved to lowest order in conformal perturbation theory. Requiring the theory to have $U_q[osp(2|2)^{(1)}]$ quantum group symmetry, we calculate the S -matrix, or more appropriately the R -matrix, for the fundamental vector representation. The physical S -matrix is then obtained, up to CDD factors, by imposing the unitarity and crossing constraints. We propose that in the marginal limit this S -matrix is the exact S -matrix (in the fundamental representation) with Yangian symmetry for the $osp(2|2)$ current-current model. The particle spectrum of the model and the corresponding S -matrix are massive. In particular, this means that all states are Anderson localized.

The quantum Hall disordered system discussed in [27] contains three types of randomness. Though the generic case, with all types of randomness present, is not believed to be integrable, it is certainly possible that on some submanifold in the three coupling parameter space the model can be exactly solved. Various such subspaces have already been investigated [27–29]. An interesting subset was recently studied in [30] (see also [31]), where supersymmetric disorder averaging led to a $gl(N|N)$ current-current type model for which exact correlation functions were computed. Our S -matrix analysis considers the situation where there is only one specific type of disorder, namely a random scalar potential.

We present our results as follows. In section 2 we write down the models and show how the full current algebra is generated under renormalization. The non-local charges are constructed in section 3. From the quantum group structure we build the S -matrix for the fundamental representation in section 4. The pole structure is briefly discussed in section 5. Lastly we conclude with a summary and comment on open questions for further study. An appendix reviews the $osp(2|2)$ algebras.

2. The $osp(2|2)$ field theory models

2a. The $osp(2|2)$ current-current model

The model introduced by Ludwig *et al.* in connection with the integer quantum Hall transition consists of a Dirac fermion $(\psi_{\pm}, \bar{\psi}_{\pm})$ coupled to, in the most general case, three types of randomness: a random vector potential A , a random mass m , and a random scalar potential V . The Euclidean action takes the

form $(z = t + ix)$

$$S = \int \frac{d^2x}{2\pi} \left(\psi_- (\partial_{\bar{z}} - iA_{\bar{z}}) \psi_+ + \bar{\psi}_- (\partial_z - iA_z) \bar{\psi}_+ + i \frac{m(x)}{2} (\bar{\psi}_- \psi_+ - \psi_- \bar{\psi}_+) + i \frac{V(x)}{2} (\bar{\psi}_- \psi_+ + \psi_- \bar{\psi}_+) \right). \quad (2.1)$$

The model we are interested in only contains a random scalar potential ($A = m = 0$), which is taken to have a gaussian distribution with mean zero and positive variance g_V

$$P[V] = \exp \left(-\frac{1}{4g_V} \int \frac{d^2x}{2\pi} V(x)^2 \right). \quad (2.2)$$

Using the supersymmetric method and averaging over $V(x)$ leads to the effective action [28]

$$S_{\text{eff}} = S_{\text{cft}} + \frac{g_V}{4} \int \frac{d^2x}{2\pi} \Phi_V \quad (2.3)$$

$$S_{\text{cft}} = \int \frac{d^2x}{2\pi} (\psi_- \partial_{\bar{z}} \psi_+ + \bar{\psi}_- \partial_z \bar{\psi}_+ + \beta \partial_{\bar{z}} \gamma + \bar{\beta} \partial_z \bar{\gamma}) \quad (2.4)$$

$$\Phi_V = (\bar{\psi}_- \psi_+ + \psi_- \bar{\psi}_+ + \bar{\beta} \gamma + \bar{\gamma} \beta)^2, \quad (2.5)$$

where $(\beta, \gamma, \bar{\beta}, \bar{\gamma})$ are complex bosonic ghosts of conformal dimensions

$$[\beta] = (\frac{1}{2}, 0), \quad [\gamma] = (\frac{1}{2}, 0), \quad [\bar{\beta}] = (0, \frac{1}{2}), \quad [\bar{\gamma}] = (0, \frac{1}{2}), \quad (2.6)$$

and satisfy the operator product expansions (OPE's)

$$\gamma(z) \beta(w) \sim -\beta(z) \gamma(w) \sim \frac{1}{z-w}. \quad (2.7)$$

The fermions satisfy the usual OPE's

$$\psi_+(z) \psi_-(w) \sim \psi_-(z) \psi_+(w) \sim \frac{1}{z-w}. \quad (2.8)$$

The action S_{cft} is anti-hermitian and has a central charge of zero, $c = 0$, since $c_{\beta, \gamma} = -1$. One can view S_{eff} as a perturbed (non-unitary) conformal field theory (CFT). Note that the perturbation is marginal (Φ_V has dimension 2). We will see below that the perturbation is actually marginally relevant.

As in the random bond Ising model [28, 32, 33], the following supersymmetric current algebra is a symmetry of the conformal action S_{cft}

$$\begin{aligned} G_{\pm}(z) &= \beta(z) \psi_{\pm}(z), & \hat{G}_{\pm}(z) &= \gamma(z) \psi_{\pm}(z) \\ K(z) &=: \beta^2(z) :, & \hat{K}(z) &=: \gamma^2(z) : \\ J(z) &=: \psi_+(z) \psi_-(z) :, & H(z) &=: \gamma(z) \beta(z) : \end{aligned} \quad (2.9)$$

$$\begin{aligned} \bar{G}_{\pm}(\bar{z}) &= \pm \bar{\beta}(\bar{z}) \bar{\psi}_{\pm}(\bar{z}), & \bar{\hat{G}}_{\pm}(\bar{z}) &= \mp \bar{\gamma}(\bar{z}) \bar{\psi}_{\pm}(\bar{z}) \\ \bar{K}(\bar{z}) &=: -\bar{\beta}^2(\bar{z}) :, & \bar{\hat{K}}(\bar{z}) &=: -\bar{\gamma}^2(\bar{z}) : \\ \bar{J}(\bar{z}) &=: \bar{\psi}_+(\bar{z}) \bar{\psi}_-(\bar{z}) :, & \bar{H}(\bar{z}) &=: \bar{\gamma}(\bar{z}) \bar{\beta}(\bar{z}) :. \end{aligned} \quad (2.10)$$

(In this paper we use the term supersymmetry merely to mean a symmetry algebra/transformation based on some Lie superalgebra and not space-time supersymmetry in the usual sense.) Here $: \dots :$ denotes normal ordering, namely $: A(w) B(w) :$ is the coefficient of $(z-w)^0$ in the OPE $A(z) B(w)$. These currents form a level one representation of the affine $osp(2|2)^{(1)}$ current algebra. The rank two $osp(2|2)$ Lie superalgebra has six roots, four odd or “fermionic” and two even or “bosonic”. The G ’s, being fermionic, are associated with the fermionic roots and the K ’s with the bosonic roots. The two simple roots, (α_1, α_2) can be chosen to be both fermionic or one fermionic and one bosonic. We will choose a purely fermionic simple root system.

With this choice the additional affine root, $\alpha_0 = -(\alpha_1 + \alpha_2)$, is associated with $K(z)$ (or $\hat{K}(z)$). (The $osp(2|2)$ algebras are discussed in the appendix.) The non-trivial OPE's for the currents are

$$\begin{aligned}
J(z)J(w) &\sim \frac{1}{(z-w)^2}, & H(z)H(w) &\sim \frac{-1}{(z-w)^2} \\
J(z)G_{\pm}(w) &\sim \frac{\pm 1}{(z-w)}G_{\pm}(w), & J(z)\hat{G}_{\pm}(w) &\sim \frac{\pm 1}{(z-w)}\hat{G}_{\pm}(w) \\
H(z)G_{\pm}(w) &\sim \frac{1}{(z-w)}G_{\pm}(w), & H(z)\hat{G}_{\pm}(w) &\sim \frac{-1}{(z-w)}\hat{G}_{\pm}(w) \\
H(z)K(w) &\sim \frac{2}{(z-w)}, & H(z)\hat{K}(w) &\sim \frac{-2}{(z-w)} \\
\hat{G}_{\pm}(z)G_{\mp}(w) &\sim \frac{1}{(z-w)^2} + \frac{1}{(z-w)}(H(w) \pm J(w)) \\
\hat{K}(z)K(w) &\sim \frac{2}{(z-w)^2} + \frac{4}{(z-w)}H(w) \\
G_{-}(z)G_{+}(w) &\sim \frac{1}{(z-w)}K(w), & \hat{G}_{-}(z)\hat{G}_{+}(w) &\sim \frac{1}{(z-w)}\hat{K}(w) \\
K(z)\hat{G}_{\pm}(w) &\sim \frac{-2}{(z-w)}G_{\pm}(w), & \hat{K}(z)G_{\pm}(w) &\sim \frac{2}{(z-w)}\hat{G}_{\pm}(w),
\end{aligned} \tag{2.11}$$

and similarly for the anti-holomorphic currents. We want to emphasize that even though some of the anti-holomorphic currents (2.10) differ from their holomorphic counterparts (2.9) by signs (e.g. $K =: \beta^2$: but $\bar{K} = -: \bar{\beta}^2$:), their OPE's are nevertheless the same, i.e., the usual OPE's obtained from (2.11) by replacing all operators $\mathcal{O}(z)$ by $\bar{\mathcal{O}}(\bar{z})$.

The field Φ_V can now be written as

$$\Phi_V = -2 \left[\bar{J}J - \bar{H}H + \frac{1}{2}(\bar{K}\hat{K} + \bar{\hat{K}}K) + \bar{G}_{-}\hat{G}_{+} - \bar{\hat{G}}_{-}G_{+} + \bar{G}_{+}\hat{G}_{-} - \bar{\hat{G}}_{+}G_{-} \right], \tag{2.12}$$

which is of the current-current form (see appendix). The interaction is thus a current-current perturbation that preserves the $osp(2|2)$ symmetry of S_{cft} . Furthermore, this implies that S_{eff} has Yangian symmetry [28]. Any S -matrix we construct must respect this Yangian symmetry.

The operator Φ_V alone forms a closed algebra. Its OPE can be calculated using (2.7) and (2.8) (or (2.11) and its anti-holomorphic version), and is found to be

$$\Phi_V(z, \bar{z})\Phi_V(w, \bar{w}) \sim -\frac{8}{|z-w|^2}\Phi_V(w, \bar{w}). \tag{2.13}$$

This leads to the beta function (to lowest order)

$$\beta_g = \frac{dg}{d \log R} = g^2, \tag{2.14}$$

where R is a length scale (see section 2b) and henceforth we drop the V subscript on g_V , writing $g \equiv g_V$. We see that the perturbation is marginally relevant: g (which is positive by definition) increases at large distances. The theory is asymptotically free in the UV. The model S_{eff} is thus in a massive regime and the S -matrix we calculate will describe scattering of massive particles. The behaviour here is opposite to that of the random bond Ising/random mass model. The random mass model ($A = V = 0$) perturbing field Φ_M , with coupling g_M , can also be written as a current-current perturbation [28]. This requires redefining four of the $osp(2|2)$ currents as

$$\bar{K} \rightarrow -\bar{K}, \quad \bar{\hat{K}} \rightarrow -\bar{\hat{K}}, \quad \bar{G}_{-} \rightarrow -\bar{G}_{-}, \quad \bar{\hat{G}}_{+} \rightarrow -\bar{\hat{G}}_{+}, \tag{2.15}$$

with all other currents unchanged. The operator algebra is unchanged under this transformation. One finds that

$$\Phi_M(z, \bar{z}) = -\Phi_V(z, \bar{z}). \quad (2.16)$$

Due to the minus sign, the beta function changes sign

$$\beta_{g_M} = -g_M^2, \quad (2.17)$$

giving a marginally irrelevant perturbation that is asymptotically free in the infra-red. In this case one expects to flow to a massless regime.

2b. The $osp(2|2)^{(1)}$ Toda-type model

One can try to construct the Yangian charges for (2.3) and then use these to build the S -matrix. However we will follow an alternative approach along the lines of [34], which though being less direct, is easier to apply.

In [34] it is shown that the Yangian symmetry associated with a current-current perturbation of the WZW model (based on a simply laced Lie algebra) can be extracted (as a marginal limit) from a “smaller” model. This smaller model, which is simply a Toda system with imaginary coupling, contains only current-current operators corresponding to the affine simple roots. The reason for working with this smaller model is that one can construct an affine quantum group (q -deformed) structure, using the method of non-local charges, only for the Toda system. The full current-current perturbation is recovered through renormalization of the Toda model (see below). From the affine quantum group symmetry the S -matrix can be constructed. The marginal limit of the Toda S -matrix then leads to the Yangian symmetric S -matrix for the original current-current model.

We now apply the same reasoning to the $osp(2|2)$ model. As in the Lie algebra case, quantum group charges cannot be constructed for the full model (2.3). A smaller model which does have quantum group symmetry, as we will show in the next section, consists of taking only half the terms in (2.12). Its action takes the form

$$\tilde{S} = S_{\text{cft}} + \frac{g}{4} \int \frac{d^2x}{2\pi} \tilde{\Phi}_V, \quad (2.18)$$

with S_{cft} as in (2.4) and

$$\tilde{\Phi}_V = -2 \left[\frac{1}{2} \overline{K} K + \overline{G}_- \hat{G}_+ + \overline{G}_+ \hat{G}_- \right]. \quad (2.19)$$

The current terms retained correspond to the affine simple roots of $osp(2|2)^{(1)}$. We have also dropped the Cartan terms $\overline{J}J$ and $\overline{H}H$. Thus (2.18) is the supersymmetric analog of the Toda system used in [34] to study current-current perturbations of the WZW model. (In order to construct the quantum charges one needs to introduce an additional parameter $\hat{\beta}$ in (2.19), which serves to make the perturbation relevant. The model (2.18) is then understood as the marginal limit $\hat{\beta} \rightarrow 1$ of the deformed model. This will become clearer in the next section.)

Unlike the full model, \tilde{S} is not renormalizable. The reason being that the operators in (2.19) do not form a closed algebra by themselves. More generally, consider a marginal perturbation of a CFT by some set of operators $\{\mathcal{O}^i(z, \bar{z})\}$

$$S = S_{\text{cft}} + \sum_i \int d^2x g_i \mathcal{O}^i(z, \bar{z}). \quad (2.20)$$

For (2.20) to be renormalizable, the operator algebra for the set $\{\mathcal{O}^i\}$ must close onto itself. That is to say the OPE's must be of the form

$$\mathcal{O}^i(z, \bar{z}) \mathcal{O}^j(w, \bar{w}) \sim C_k^{ij} \frac{1}{|z-w|^2} \mathcal{O}^k(w, \bar{w}), \quad (2.21)$$

for some structure constants C_k^{ij} , with any operators appearing in (2.21) already being present in the action (2.20). In this case Zamolodchikov [35] has shown that the beta functions to lowest order (1 loop) are

$$\beta_{g_i} = \frac{dg_i}{d \log R} = -\pi C_i^{jk} g_j g_k. \quad (2.22)$$

Specializing to (2.18), the action can be written as

$$\tilde{S} = S_{\text{cft}} + \sum_{i=1}^3 g_i \int d^2x \mathcal{O}^i, \quad (2.23)$$

where

$$\mathcal{O}^1 = \overline{G}_- \widehat{G}_+, \quad \mathcal{O}^2 = \overline{G}_+ \widehat{G}_-, \quad \mathcal{O}^3 = \overline{\widehat{K}} K, \quad (2.24)$$

and we have allowed the couplings to be independent. Running the renormalization procedure, one finds that after the first iteration the following operators are generated

$$\mathcal{O}^4 = \overline{\widehat{G}}_+ G_-, \quad \mathcal{O}^5 = \overline{\widehat{G}}_- G_+, \quad \mathcal{O}^6 = \overline{K} \widehat{K}. \quad (2.25)$$

These correspond to the negative simple roots. However, the operator algebra for $\{\mathcal{O}^i\}_{1 \leq i \leq 6}$ still does not close. A second run generates the Cartan operators

$$\mathcal{O}^7 = (\overline{H} + \overline{J})(H + J), \quad \mathcal{O}^8 = (\overline{H} - \overline{J})(H - J), \quad \mathcal{O}^9 = \overline{H} H. \quad (2.26)$$

The operator algebra for $\{\mathcal{O}^i\}_{1 \leq i \leq 9}$ closes, and the resulting action is

$$\tilde{S} = S_{\text{cft}} + \int d^2x \sum_{i=1}^9 g_i \mathcal{O}^i. \quad (2.27)$$

The beta functions are found using (2.22) to be

$$\begin{aligned} \beta_{g_1} &= 2\pi g_1(4g_8 + g_9) + 8\pi g_5 g_6, & \beta_{g_2} &= 2\pi g_2(4g_7 + g_9) + 8\pi g_4 g_6 \\ \beta_{g_3} &= 8\pi g_3(g_7 + g_8 + g_9) + 2\pi g_4 g_5 \\ \beta_{g_4} &= 2\pi g_4(4g_8 + g_9) + 8\pi g_2 g_3, & \beta_{g_5} &= 2\pi g_5(4g_7 + g_9) + 8\pi g_1 g_3 \\ \beta_{g_6} &= 8\pi g_6(g_7 + g_8 + g_9) + 2\pi g_1 g_2 \\ \beta_{g_7} &= 2\pi g_1 g_4, & \beta_{g_8} &= 2\pi g_2 g_5 \\ \beta_{g_9} &= 32\pi g_3 g_6. \end{aligned} \quad (2.28)$$

We can easily verify that a solution to (2.28) exists which gives the original model (2.3) and its beta function (2.14). Since (2.3) contains only one coupling, we should try the ansatz $g_i = -\frac{1}{4\pi} \alpha_i g$, where α_i are some numerical constants. (We remark that the α_i 's do not spoil the quantum group symmetry and one can work with (2.23) using this ansatz instead of (2.18).) It is straight-forward to check that the set

$$\{\alpha_1 = \alpha_2 = 1, \alpha_3 = \frac{1}{2}, \alpha_4 = \alpha_5 = -1, \alpha_6 = \frac{1}{2}, \alpha_7 = \alpha_8 = \frac{1}{2}, \alpha_9 = -2\}, \quad (2.29)$$

reproduces (2.3) and reduces (2.28) to the single beta function (2.14) for g . Thus under renormalization (2.18) leads to the $osp(2|2)$ current-current model. We will therefore construct the S -matrix for the Toda-type system (2.18) and argue that it gives the required S -matrix for (2.3) in the marginal limit.

3. Quantum group symmetry in the $osp(2|2)$ models

3a. Non-local charges and conformal perturbation theory

Quantum group symmetry is realized by non-local charges [34, 36–38] constructed using conformal perturbation theory. We outline the main points of the construction below. (For more details see [34].)

Suppose we have a CFT perturbed by a relevant spin-zero field

$$\Phi_{\text{pert.}}(z, \bar{z}) = \phi_{\text{pert.}}(z) \overline{\phi}_{\text{pert.}}(\bar{z}), \quad (3.1)$$

with the Euclidean action

$$S = S_{\text{cft}} + g \int \frac{d^2x}{2\pi} \Phi_{\text{pert.}}(z, \bar{z}), \quad (3.2)$$

and some currents $\{J^a, \bar{J}^{\bar{a}}\}$ which are chiral when $g = 0$

$$\partial_z J^a = \partial_{\bar{z}} \bar{J}^{\bar{a}} = 0 \implies J^a = J^a(z), \quad \bar{J}^{\bar{a}} = \bar{J}^{\bar{a}}(\bar{z}). \quad (3.3)$$

In the perturbed theory (3.2) these currents are no longer chiral, but to lowest order in g satisfy the following equations of motion [39]

$$\begin{aligned} \partial_{\bar{z}} J^a(z, \bar{z}) &= g \oint_z \frac{d^2w}{2\pi i} \Phi_{\text{pert.}}(w, \bar{z}) J^a(z) \\ \partial_z \bar{J}^{\bar{a}}(z, \bar{z}) &= g \oint_{\bar{z}} \frac{d^2\bar{w}}{2\pi i} \Phi_{\text{pert.}}(z, \bar{w}) \bar{J}^{\bar{a}}(\bar{z}). \end{aligned} \quad (3.4)$$

If the residues of the OPE's on the righthand side are total derivatives

$$\text{Res}_{z=w}(\phi_{\text{pert.}}(w) J^a(z)) = \partial_z h^a(z), \quad \text{Res}_{\bar{z}=\bar{w}}(\bar{\phi}_{\text{pert.}}(\bar{w}) \bar{J}^{\bar{a}}(\bar{z})) = \partial_{\bar{z}} \bar{h}^{\bar{a}}(\bar{z}), \quad (3.5)$$

then (3.4) becomes

$$\partial_{\bar{z}} J^a(z, \bar{z}) = \partial_z H^a(z, \bar{z}), \quad \partial_z \bar{J}^{\bar{a}}(z, \bar{z}) = \partial_{\bar{z}} \bar{H}^{\bar{a}}(z, \bar{z}), \quad (3.6)$$

where

$$H^a(z, \bar{z}) = g h^a(z) \bar{\phi}_{\text{pert.}}(\bar{z}), \quad \bar{H}^{\bar{a}}(z, \bar{z}) = g \bar{h}^{\bar{a}}(\bar{z}) \phi_{\text{pert.}}(z). \quad (3.7)$$

The equations of motion (3.6) imply that to lowest order in g we have the conserved charges

$$\begin{aligned} Q^a &= \frac{1}{2\pi i} \left(\int dz J^a + \int d\bar{z} H^a \right) \\ \bar{Q}^{\bar{a}} &= \frac{1}{2\pi i} \left(\int d\bar{z} \bar{J}^{\bar{a}} + \int dz \bar{H}^{\bar{a}} \right). \end{aligned} \quad (3.8)$$

The currents $\{J^a, H^a\}$ and $\{\bar{J}^{\bar{a}}, \bar{H}^{\bar{a}}\}$, and hence the charges $\{Q^a, \bar{Q}^{\bar{a}}\}$, are non-local. To see this consider the specific case where S_{cft} is a sum of free bosons

$$S_{\text{cft}} = \frac{1}{8\pi} \int d^2x \sum_i \partial_\mu \Phi_i \partial_\mu \Phi_i. \quad (3.9)$$

In the limit $g = 0$, Φ_i can be expanded into its chiral components, $\Phi_i(z, \bar{z}) = \phi_i(z) + \bar{\phi}_i(\bar{z})$, with

$$\langle \phi_i(z) \phi_j(w) \rangle = -\delta_{ij} \log(z - w), \quad \langle \bar{\phi}_i(\bar{z}) \bar{\phi}_j(\bar{w}) \rangle = -\delta_{ij} \log(\bar{z} - \bar{w}). \quad (3.10)$$

If $g \neq 0$, Φ_i can again be written as $\Phi_i(x, t) = \phi_i(x, t) + \bar{\phi}_i(x, t)$, but now $\phi_i(x, t)$ and $\bar{\phi}_i(x, t)$ are no longer chiral. For arbitrary g , $\phi_i(x, t)$ and $\bar{\phi}_i(x, t)$ can be written in the non-local way

$$\begin{aligned} \phi_i(x, t) &= \frac{1}{2} \left(\Phi_i(x, t) + \int_{-\infty}^x dy \partial_t \Phi_i(y, t) \right) \stackrel{g=0}{=} \phi_i(z) \\ \bar{\phi}_i(x, t) &= \frac{1}{2} \left(\Phi_i(x, t) - \int_{-\infty}^x dy \partial_t \Phi_i(y, t) \right) \stackrel{g=0}{=} \bar{\phi}_i(\bar{z}). \end{aligned} \quad (3.11)$$

Since the currents are in general functions of ϕ_i and $\bar{\phi}_i$, they are non-local due to (3.11). This non-locality leads to non-trivial braiding relations for the currents

$$J^a(x, t) \bar{J}^{\bar{a}}(y, t) = R_{bb}^{a\bar{a}} \bar{J}^{\bar{b}}(y, t) J^b(x, t), \quad (3.12)$$

where $R_{bb}^{a\bar{a}}$ is a braiding matrix which will depend on the couplings and the parities of the currents, i.e. whether the currents are even (bosonic) or odd (fermionic). The corresponding result for the charges is

$$Q^a \bar{Q}^{\bar{a}} - R_{bb}^{a\bar{a}} \bar{Q}^{\bar{b}} Q^b = T^{a\bar{a}}, \quad (3.13)$$

where $T^{a\bar{a}}$ is a topological charge

$$T^{a\bar{a}} = \frac{g}{2\pi i} \int_t (dz \partial_z + d\bar{z} \partial_{\bar{z}}) h^a(z) \bar{h}^{\bar{a}}(\bar{z}). \quad (3.14)$$

3b. Bosonization and the $\hat{\beta}$ parameters

The above formalism is valid for a relevant perturbation. However, the perturbing field (2.19) is marginal. By making the currents depend on a parameter $\hat{\beta}$, the perturbation can be made relevant, with the marginal limit being $\hat{\beta} \rightarrow 1$. In order to introduce this additional parameter the action (2.18) has to be (partially) bosonized.

The fermions can be bosonized in the standard way

$$\psi_{\pm}(z) = e^{\pm i\phi_1(z)}, \quad \bar{\psi}_{\pm}(\bar{z}) = e^{\mp i\bar{\phi}_1(\bar{z})}. \quad (3.15)$$

The bosonic ghosts can be written as [40, 41]

$$\begin{aligned} \gamma(z) &= e^{\phi_2(z)} \eta(z), & \bar{\gamma}(\bar{z}) &= e^{-\bar{\phi}_2(\bar{z})} \bar{\eta}(\bar{z}) \\ \beta(z) &= e^{-\phi_2(z)} \partial \xi(z), & \bar{\beta}(\bar{z}) &= e^{\bar{\phi}_2(\bar{z})} \bar{\partial} \bar{\xi}(\bar{z}). \end{aligned} \quad (3.16)$$

The chiral bosons $\{\phi_i(z), \bar{\phi}_i(\bar{z})\}$ satisfy (3.10), and $\{\eta(z), \xi(z)\}$ and $\{\bar{\eta}(\bar{z}), \bar{\xi}(\bar{z})\}$ are fermionic ghost systems with conformal dimensions and OPE's

$$[\eta] = (1, 0), \quad [\xi] = (0, 0), \quad [\bar{\eta}] = (0, 1), \quad [\bar{\xi}] = (0, 0) \quad (3.17)$$

$$\eta(z) \xi(w) \sim \xi(z) \eta(w) \sim \frac{1}{(z-w)}, \quad \bar{\eta}(\bar{z}) \bar{\xi}(\bar{w}) \sim \bar{\xi}(\bar{z}) \bar{\eta}(\bar{w}) \sim \frac{1}{(\bar{z}-\bar{w})}. \quad (3.18)$$

The central charge for the ghost system, $c_{\eta, \xi}$, is -2 . The $osp(2|2)^{(1)}$ currents expressed in terms of the new fields become

$$\begin{aligned} J &= i\partial\phi_1, & \bar{J} &= -i\bar{\partial}\bar{\phi}_1 \\ H &= \partial\phi_2, & \bar{H} &= -\bar{\partial}\bar{\phi}_2 \\ G_{\pm} &= \exp\left(i\alpha_{1,2} \cdot \vec{\phi}\right) \partial\xi, & \bar{G}_{\pm} &= \pm \exp\left(-i\alpha_{1,2} \cdot \vec{\bar{\phi}}\right) \bar{\partial}\bar{\xi} \\ \hat{G}_{\pm} &= \exp\left(-i\alpha_{2,1} \cdot \vec{\phi}\right) \eta, & \bar{\hat{G}}_{\pm} &= \mp \exp\left(i\alpha_{2,1} \cdot \vec{\bar{\phi}}\right) \bar{\eta} \\ K &= \exp\left(-i\alpha_0 \cdot \vec{\phi}\right) : \partial^2 \xi \partial \xi :, & \bar{K} &= -\exp\left(i\alpha_0 \cdot \vec{\bar{\phi}}\right) : \bar{\partial}^2 \bar{\xi} \bar{\partial} \bar{\xi} : \\ \hat{K} &= \exp\left(i\alpha_0 \cdot \vec{\phi}\right) : \partial \eta \eta :, & \bar{\hat{K}} &= -\exp\left(-i\alpha_0 \cdot \vec{\bar{\phi}}\right) : \bar{\partial} \bar{\eta} \bar{\eta} :, \end{aligned} \quad (3.19)$$

where

$$\vec{\phi} = (\phi_1, \phi_2), \quad \vec{\bar{\phi}} = (\bar{\phi}_1, \bar{\phi}_2), \quad \vec{\Phi} = (\Phi_1, \Phi_2), \quad (3.20)$$

and we have introduced the simple roots for $osp(2|2)^{(1)}$ (see appendix)

$$\alpha_1 = (1, i), \quad \alpha_2 = (-1, i), \quad \alpha_0 = (0, -2i). \quad (3.21)$$

In deriving the expressions for H, K and \hat{K} , we have made use of

$$e^{\pm\phi_2(z)} e^{\mp\phi_2(w)} = (z-w)(1 \pm \partial_w \phi_2(w)(z-w) + \dots) \quad (3.22)$$

$$e^{\pm\phi_2(z)}e^{\pm\phi_2(w)} = \frac{1}{(z-w)}e^{\pm 2\phi_2(w)} + \dots \quad (3.23)$$

$$\eta(z)\eta(w) = (z-w)(:\partial_w\eta(w)\eta(w): + \dots) \quad (3.24)$$

$$\partial_z\xi(z)\partial_w\xi(w) = (z-w)(:\partial_w^2\xi(w)\partial_w\xi(w): + \dots) \quad (3.25)$$

$$:\eta(w)\eta(w): = :\partial_w\xi(w)\partial_w\xi(w): = 0, \quad (3.26)$$

and similarly for the anti-holomorphic parts. One can check that (3.19) satisfy the required OPE's (2.11).

Now we can write the perturbed CFT which renormalizes to (2.3) at the marginal point. The required action takes the form

$$\tilde{S}^d = S_{\text{cft}} + S_{\text{pert.}}^d \quad (3.27)$$

$$S_{\text{cft}} = \frac{1}{4\pi} \int d^2x \left(\frac{1}{2}(\partial_\mu\Phi_1)^2 + \frac{1}{2}(\partial_\mu\Phi_2)^2 + \eta\bar{\partial}\xi + \bar{\xi}\partial\eta \right) \quad (3.28)$$

$$S_{\text{pert.}}^d = -\frac{g}{4} \int d^2x \left(e^{-i\hat{\beta}\alpha_1\cdot\vec{\Phi}}\bar{\partial}\bar{\xi}\eta - e^{-i\hat{\beta}\alpha_2\cdot\vec{\Phi}}\bar{\partial}\bar{\xi}\eta - \frac{1}{2}e^{-i\hat{\beta}\alpha_0\cdot\vec{\Phi}} : \bar{\partial}\eta\eta : : \partial^2\xi\partial\xi : \right). \quad (3.29)$$

At the marginal point, $\hat{\beta} = 1$, \tilde{S}^d gives the bosonized version of \tilde{S} . The beta parameter is analogous to the β (or $\hat{\beta}$) parameter in the sine-Gordon model [7, 34], and can be thought of as arising due to an anisotropy in the current-current perturbation. If we add to (2.18) the term

$$\frac{1}{8\pi} \int d^2x \rho (\bar{J}J - \bar{H}H) = \frac{1}{8\pi} \int d^2x (\rho(\partial_\mu\Phi_1)^2 + \rho(\partial_\mu\Phi_2)^2), \quad (3.30)$$

and rescale $\Phi_i \rightarrow \Phi_i/\sqrt{1+\rho}$, one arrives at (3.27) with $\hat{\beta} = 1/\sqrt{1+\rho}$. In view of this one can interpret \tilde{S}^d as a deformation of (2.18). Indeed we will show that the beta parameter is related to the quantum group deformation parameter. The Toda-type structure is clearly seen in the bosonized form (3.29). In comparison with the Lie algebraic Toda models, the main difference here is the inclusion of fermionic fields due to supersymmetry.

For arbitrary positive values of $\hat{\beta}$, the three terms in $S_{\text{pert.}}^d$ do not have the same dimensions. The first two terms are marginal for all values of $\hat{\beta}$, whereas the remaining $\bar{K}K$ term is relevant for $\hat{\beta} > 1$. This suggests that $S_{\text{pert.}}^d$ should be rewritten with two couplings, g_1 and g_2 , with one of them being dimensionless, which in turn seems to imply that conformal perturbation theory cannot be used to construct the non-local charges. However, these problems can be resolved by introducing a background charge \mathcal{Q} coupled to the field Φ_2 [41]. The conformal dimensions of $\exp(i\alpha\phi_2)$ and $\exp(i\alpha\bar{\phi}_2)$ are then changed from $\alpha^2/2$ to $\alpha(\alpha - \mathcal{Q})/2$. One finds that for a purely imaginary charge of

$$\mathcal{Q} = i\frac{4}{3}\frac{\hat{\beta}^2 - 1}{\hat{\beta}}, \quad (3.31)$$

all three terms take of the same conformal dimensions and giving

$$\Delta(g) = \bar{\Delta}(g) = \frac{2}{3}(\hat{\beta}^2 - 1). \quad (3.32)$$

Therefore the perturbation as it is written in (3.29), with one coupling g , is consistent and can be treated as being relevant for $\hat{\beta} > 1$. Since introducing \mathcal{Q} does not change the OPE's, the construction of the non-local charges is identical for both cases $\mathcal{Q} = 0$ and $\mathcal{Q} \neq 0$. We choose to work with $\mathcal{Q} = 0$. All expressions that follow, in particular the quantum group symmetry equations, also hold for $\mathcal{Q} \neq 0$. The only difference between $\mathcal{Q} = 0$ and $\mathcal{Q} \neq 0$ is that the spins of the charges are changed. This means that in going from $\mathcal{Q} = 0$ to $\mathcal{Q} \neq 0$, the gradation of the quantum affine algebra changes from the homogeneous to the principal gradation (see below). Thus by taking $\mathcal{Q} = 0$ we are effectively working in the homogeneous gradation.

3c. The quantum group charges

In this section we construct the currents $\{J^a, H^a\}$ and $\{\bar{J}^{\bar{a}}, \bar{H}^{\bar{a}}\}$, and the charges $\{Q^a, \bar{Q}^{\bar{a}}\}$, satisfying the conservation laws (3.6) for the theory (3.27).

Taking into account (i) the Toda-type structure of (3.29) and (ii) the forms of the known currents for non-supersymmetric Toda [34], one expects the J^a 's to be some combinations of the vertex operators with the fermions

$$J^a \sim \exp(ia_1\phi_1 + ia_2\phi_2) \times \{\eta, \partial\xi, \partial\eta\eta, \partial^2\xi\xi\}, \quad (3.33)$$

and similarly for the $\bar{J}^{\bar{a}}$'s. We find the following currents

$$\begin{aligned} J^{1,2} &= \exp\left(\frac{i}{\beta}\alpha_{1,2} \cdot \vec{\phi}\right) \partial\xi \\ J^0 &= \exp\left(\frac{i}{\beta}\alpha_0 \cdot \vec{\phi}\right) : \partial\eta\eta : \\ \bar{J}^{1,2} &= \exp\left(\frac{i}{\beta}\alpha_{1,2} \cdot \vec{\bar{\phi}}\right) \bar{\eta} \\ \bar{J}^0 &= \exp\left(\frac{i}{\beta}\alpha_0 \cdot \vec{\bar{\phi}}\right) : \bar{\partial}^2\xi \partial\bar{\xi} : . \\ H^{1,2} &= \rho_{1,2}^{(0)} g\hat{\gamma} \exp\left(-i\frac{\hat{\beta}}{\hat{\gamma}}\alpha_{1,2} \cdot \vec{\phi} - i\hat{\beta}\alpha_{1,2} \cdot \vec{\bar{\phi}}\right) \partial\bar{\xi} \\ H^0 &= \rho_0^{(0)} g\hat{\gamma} \exp\left(-i\frac{\hat{\beta}}{\hat{\gamma}}\alpha_0 \cdot \vec{\phi} - i\hat{\beta}\alpha_0 \cdot \vec{\bar{\phi}}\right) : \partial\bar{\eta}\bar{\eta} : \\ \bar{H}^{1,2} &= \rho_{1,2}^{(0)} g\hat{\gamma} \exp\left(-i\frac{\hat{\beta}}{\hat{\gamma}}\alpha_{1,2} \cdot \vec{\bar{\phi}} - i\hat{\beta}\alpha_{1,2} \cdot \vec{\phi}\right) \eta \\ \bar{H}^0 &= \rho_0^{(0)} g\hat{\gamma} \exp\left(-i\frac{\hat{\beta}}{\hat{\gamma}}\alpha_0 \cdot \vec{\bar{\phi}} - i\hat{\beta}\alpha_0 \cdot \vec{\phi}\right) : \partial^2\xi \partial\xi : , \end{aligned} \quad (3.34)$$

$$\quad (3.35)$$

where

$$\frac{1}{\hat{\gamma}} \equiv 1 - \frac{1}{\hat{\beta}^2} \geq 0. \quad (3.36)$$

The $\rho^{(0)}$'s are numerical constants that depend on (i) the coefficients of the terms in (3.29) and (ii) the roots α_i 's. In the following, similar numerical factors will be denoted as $\rho_i^{(k)}$. These factors are normalization factors and their exact values are not needed in establishing quantum group symmetry and calculation of the S -matrix. A reminder that the fields $\vec{\phi}(x, t)$ and $\vec{\bar{\phi}}(x, t)$ are given by the non-local expressions (3.11). These currents obey the conservation laws

$$\bar{\partial}J^i = \partial H^i, \quad \partial\bar{J}^i = \bar{\partial}\bar{H}^i \quad (i = 1, 2, 3). \quad (3.37)$$

The corresponding conserved (to lowest order) charges are

$$Q^i = \frac{1}{2\pi i} \left(\int dz J^i + \int d\bar{z} H^i \right) \quad (3.38a)$$

$$\bar{Q}^i = \frac{1}{2\pi i} \left(\int d\bar{z} \bar{J}^i + \int dz \bar{H}^i \right). \quad (3.38b)$$

The Lorentz spins s of the charges, defined as $s = \Delta - \bar{\Delta}$, are

$$\begin{aligned} s(Q^{1,2}) &= s(\bar{Q}^{1,2}) = 0 \\ s(Q^0) &= -s(\bar{Q}^0) = \frac{2}{\hat{\gamma}}. \end{aligned} \quad (3.39)$$

Note that if we included the background charge \mathcal{Q} (3.31), then all Q^i (or \overline{Q}^i) would take on the same Lorentz spin of

$$s(Q^i) = -s(\overline{Q}^i) = \frac{1}{3} \frac{2}{\hat{\gamma}}, \quad i = 0, 1, 2. \quad (3.40)$$

The charges are also assigned parities $d_i, \overline{d}_i \in \{0, 1\}$, determined by the number of fermionic fields (i.e. the η 's and ξ 's) they contain. Since $Q^{1,2}$ and $\overline{Q}^{1,2}$ contain an odd number of fermions, they are referred to as being odd or fermionic with $d_{1,2} = \overline{d}_{1,2} = 1$. The even or bosonic charges Q^0 and \overline{Q}^0 , consisting of an even number of fermions, have $d_0 = \overline{d}_0 = 0$. All the currents have the same parities as the corresponding charges. In general, the parity of an operator \mathcal{O} will be denoted $d(\mathcal{O})$.

To obtain the algebra satisfied by the charges, we need the braiding relations for the currents. These can be found using

$$\exp(ia\alpha_i \cdot \vec{\phi}(x, t)) \exp(ib\alpha_j \cdot \vec{\phi}(y, t)) = e^{\pm iab\pi\alpha_i \cdot \alpha_j} \exp(ib\alpha_j \cdot \vec{\phi}(y, t)) \exp(ia\alpha_i \cdot \vec{\phi}(x, t)), \quad x \gtrless y \quad (3.41a)$$

$$\exp(ia\alpha_i \cdot \vec{\vec{\phi}}(x, t)) \exp(ib\alpha_j \cdot \vec{\vec{\phi}}(y, t)) = e^{\mp iab\pi\alpha_i \cdot \alpha_j} \exp(ib\alpha_j \cdot \vec{\vec{\phi}}(y, t)) \exp(ia\alpha_i \cdot \vec{\vec{\phi}}(x, t)), \quad x \gtrless y \quad (3.41b)$$

$$\exp(ia\alpha_i \cdot \vec{\phi}(x, t)) \exp(ib\alpha_j \cdot \vec{\vec{\phi}}(y, t)) = e^{iab\pi\alpha_i \cdot \alpha_j} \exp(ib\alpha_j \cdot \vec{\vec{\phi}}(y, t)) \exp(ia\alpha_i \cdot \vec{\phi}(x, t)), \quad \forall x, y. \quad (3.41c)$$

To derive (3.41) one makes use of the non-local expressions (3.11) and the canonical commutation relations

$$[\Phi_i(x, t), \partial_t \Phi_j(y, t)] = \delta_{ij} 4\pi i \delta(x - y). \quad (3.42)$$

From (3.41) one gets

$$J^i(x, t) \overline{J}^j(y, t) = (-1)^{d_i \overline{d}_j} e^{i \frac{\pi}{\beta^2} \alpha_i \cdot \alpha_j} \overline{J}^j(y, t) J^i(x, t), \quad \forall x, y \quad (3.43a)$$

$$H^i(x, t) \overline{H}^j(y, t) = (-1)^{d_i \overline{d}_j} e^{i \frac{\pi}{\beta^2} \alpha_i \cdot \alpha_j} \overline{H}^j(y, t) H^i(x, t), \quad \forall x, y. \quad (3.43b)$$

Since (3.43b) involves applying (3.41a) and (3.41b), its validity for all x and y can be shown using a limiting procedure. The relations (3.43), combined with (3.13) and (3.14), imply that the charges satisfy

$$Q^i \overline{Q}^j - (-1)^{d_i \overline{d}_j} e^{i \frac{\pi}{\beta^2} \alpha_i \cdot \alpha_j} \overline{Q}^j Q^i = \rho_i^{(1)} \delta_{ij} \frac{g \hat{\gamma}^2}{2\pi i} \int_t dx \partial_x \left[\exp \left(-i \frac{\hat{\beta}}{\hat{\gamma}} \alpha_i \cdot \vec{\Phi} \right) \right]. \quad (3.44)$$

The right-hand side of (3.44) can be written in terms of standard topological charges. We take a “soliton” configuration satisfying $\vec{\Phi}(x = \infty) = 0$, and define the topological charges T^i

$$T^i = \frac{\hat{\beta}}{2\pi} \int_t dx \partial_x (\alpha_i \cdot \vec{\Phi}) = -\frac{\hat{\beta}}{2\pi} \alpha_i \cdot \vec{\Phi}(x = -\infty). \quad (3.45)$$

The topological charge of any fundamental soliton field is always an integer. Also the topological charges are not all independent. In particular $T^0 = -(T^1 + T^2)$, which is a statement of the fact that the center is zero. Equation (3.44) now takes the form

$$Q^i \overline{Q}^j - (-1)^{d_i \overline{d}_j} q_1^{-\alpha_i \cdot \alpha_j} \overline{Q}^j Q^i = \rho_i^{(1)} \delta_{ij} \frac{g \hat{\gamma}^2}{2\pi i} (1 - q_1^{2T^i}), \quad (3.46)$$

where

$$q_1 = \exp(-i\pi/\hat{\beta}^2) = -\exp(i\pi/\hat{\gamma}). \quad (3.47)$$

An equivalent expression for (3.44) is

$$\overline{Q}^i Q^j - (-1)^{d_j \overline{d}_i} q_2^{-\alpha_i \cdot \alpha_j} Q^j \overline{Q}^i = -\rho_i^{(1)} \delta_{ij} (-1)^{\overline{d}_i d_i} q_2^{-\alpha_i \cdot \alpha_i} \frac{g \hat{\gamma}^2}{2\pi i} (1 - q_2^{-2T^i}), \quad (3.48)$$

where q_2 is

$$q_2 = \frac{1}{q_1} = \exp(i\pi/\hat{\beta}^2) = -\exp(-i\pi/\hat{\gamma}). \quad (3.49)$$

Since $\alpha_i \cdot \alpha_j$ is always an even integer, we can take $q_i \rightarrow -q_i$ without changing (3.46) or (3.48). The braiding relations of the topological charges with the Q 's, which are simply (undeformed) commutators, are found using

$$[T^i, \exp(i a \alpha_j \cdot \vec{\phi} + i \bar{a} \alpha_j \cdot \vec{\bar{\phi}})] = \hat{\beta}(a - \bar{a})(\alpha_i \cdot \alpha_j) \exp(i a \alpha_j \cdot \vec{\phi} + i \bar{a} \alpha_j \cdot \vec{\bar{\phi}}). \quad (3.50)$$

Equation (3.50) is most easily obtained with the complex form for T^i

$$T^i = \frac{\hat{\beta}}{2\pi} \left(\int dz \alpha_i \cdot \partial_z \vec{\phi} - \int d\bar{z} \alpha_i \cdot \partial_{\bar{z}} \vec{\bar{\phi}} \right). \quad (3.51)$$

We find the commutation relations

$$[T^i, Q^j] = \alpha_i \cdot \alpha_j Q^j \quad (3.52a)$$

$$[T^i, \bar{Q}^j] = -\alpha_i \cdot \alpha_j \bar{Q}^j \quad (3.52b)$$

$$[T^i, T^j] = 0. \quad (3.52c)$$

The topological charges are even or bosonic operators, with parities $\tilde{d}_i \equiv d(T^i) = 0$. Note that in the above braiding/commutation relations, the only purpose served by the η - ξ fermion fields is to produce the correct graded structure, namely the $(-1)^{d_i \tilde{d}_j}$ factors. Unlike the bosons ϕ_i and $\bar{\phi}_i$, the fermions are treated as being local and hence their braiding relations are simply of the form: $\psi\bar{\psi} = -\bar{\psi}\psi$.

The equations (3.46)/(3.48) and (3.52) give the symmetry algebra of the theory (3.27) to lowest order in perturbation theory. In fact, this algebra is the q -deformation of the the untwisted affine Lie superalgebra $osp(2|2)^{(1)}$, denoted $U_q[osp(2|2)^{(1)}]$, with zero center. The only relations missing above are the Serre relations for $U_q[osp(2|2)^{(1)}]$. A review of the quantum algebra $U_q[osp(2|2)^{(1)}]$, or quantum group as it is often called, is presented in the appendix. Let $e_i, f_i, h_i, i = 0, 1, 2$, be the Chevalley generators of $U_q[osp(2|2)^{(1)}]$. Using the defining relations (A.9) and (A.10), one can show that the generators satisfy

$$\left(e_i q^{h_i/2} \right) \left(f_j q^{h_j/2} \right) - (-1)^{d(e_i)d(f_j)} q^{-a_{ij}} \left(f_j q^{h_j/2} \right) \left(e_i q^{h_i/2} \right) = \delta_{ij} \frac{qq^{-a_{ii}/2}}{(1-q^2)} (1-q^{2h_i}), \quad (3.53)$$

where $a_{ij} \equiv \alpha_i \cdot \alpha_j$ is the generalized $osp(2|2)^{(1)}$ Cartan matrix. Comparing (3.53) with (3.46) or (3.48), and the commutation relations (A.9a),(A.9b) with (3.52), we can relate the non-local charges to the quantum group generators in two different ways. The first set of relations, which follow from considering (3.46) and taking $q = q_1$, are

$$Q^i = c_i e_i q^{h_i/2} \quad (d_i = d(e_i)), \quad \bar{Q}^i = c_i f_i q^{h_i/2} \quad (\bar{d}_i = d(f_i)), \quad T^i = h_i \quad (\tilde{d}_i = d(h_i)), \quad (3.54)$$

where the c_i 's satisfy

$$c_i^2 = \rho_i^{(1)} \frac{g\hat{\gamma}^2}{2\pi i} q^{-1} q^{a_{ii}/2} (1-q^2). \quad (3.55)$$

The second set, obtained from (3.48) and setting $q = q_2$, gives

$$\bar{Q}^i = c_i e_i q^{h_i/2} \quad (\bar{d}_i = d(e_i)), \quad Q^i = c_i f_i q^{h_i/2} \quad (d_i = d(f_i)), \quad T^i = -h_i \quad (\tilde{d}_i = d(h_i)), \quad (3.56)$$

with

$$c_i^2 = -\rho_i^{(1)} (-1)^{d_i \tilde{d}_i} \frac{g\hat{\gamma}^2}{2\pi i} q^{-1} q^{-a_{ii}/2} (1-q^2). \quad (3.57)$$

(We have not yet shown $(Q^{1,2})^2 = (\bar{Q}^{1,2})^2 = 0$, corresponding to the defining relation (A.9d). That this holds can be verified from the explicit expressions in the fundamental representation discussed below.) From now on we will work with the second set (3.56) and take the deformation parameter q to be

$$q = \exp(i\pi/\hat{\beta}^2) = -\exp(-i\pi/\hat{\gamma}). \quad (3.58)$$

Equation (3.58) gives the relationship between the beta parameter $\hat{\beta}$ and the quantum group deformation parameter q mentioned earlier. The marginal point ($\hat{\beta} = 1$) corresponds to $q = -1$.

Some important remarks regarding the relationship between the models (2.18) and (3.27) and the quantum group symmetry need to be made. For $\hat{\beta} > 1$, the model (3.27) has quantum group symmetry. At the marginal point (3.27) becomes (2.18) which, when fully renormalized, has Yangian symmetry rather than quantum group symmetry. This means that by carefully taking the marginal limit $\hat{\beta} \rightarrow 1$ of the algebra (3.48), the Yangian symmetry algebra can be extracted. Or at the level of S -matrices, an S -matrix that is symmetric under (3.38) and (3.45) (or (3.56)) should in the marginal limit give the Yangian symmetric S -matrix of (2.3). So we can consider the charges $\{Q^i, \bar{Q}^i, T^i\}$ as generating a symmetry of (3.27) for $\hat{\beta} > 1$, with the understanding that for the system (2.3) the limit $\hat{\beta} \rightarrow 1$ needs to be taken. The symmetry structure here is similar to that of the affine $sl(n)$ Toda model, and specifically to that of the sine-Gordon model, which has $U_q(sl(2)^{(1)})$ symmetry for any $\hat{\beta} < \sqrt{2}$ and this reduces to a Yangian symmetry as $\hat{\beta} \rightarrow \sqrt{2}$ [34]. We have obtained this symmetry algebra as a lowest order result. Showing that the algebra is exact to all orders in perturbation theory amounts to giving a scaling argument that forbids any higher order terms in g on the right side of (3.44) [39, 34]. We do not discuss exactness here since the lowest order result is sufficient to obtain the S -matrix. This is because the constraints on S placed by (i) the lowest order quantum group/Yangian symmetry and (ii) the scattering constraints of Yang-Baxter, unitarity and crossing, are restrictive enough so that higher order contributions (if any) should only be of the CDD type. In this sense the S -matrix we calculate is a “minimal” S -matrix. Lastly, in the marginal limit it may seem that the charges (3.56) blow up since $\hat{\gamma} \rightarrow +\infty$. To resolve this one can regularize the limit by also taking $g \rightarrow 0$ such that $\lim_{\hat{\beta} \rightarrow 1} (g\hat{\gamma})$ is finite. The sine-Gordon charges also need to be regularized in the marginal limit. Another model which displays this behavior is the multi-cosine model (see [42] and references therein).

3d. The fundamental fields and the comultiplication

In this section we construct the quantum fields that create the particles in the fundamental vector representation. We also determine the comultiplication of the quantum charges, that is, the action of the charges on asymptotic multiparticle states. For completeness, we begin with a brief review of the fundamental vector representation (see [43–47] for further details).

The fundamental vector representation V of $osp(2|2)$ (denoted $(0, 1/2)$ in [43]) is four dimensional with a basis $\{|1\rangle, |2\rangle, |3\rangle, |4\rangle\}$. (We will use the same symbol for the representation and the associated vector space.) Since we are dealing with a Lie superalgebra, the vector space is Z_2 -graded and the states can be assigned parities, $d(i)$, in two different ways: (i) $|1\rangle$ and $|4\rangle$ are even, $d(1) = d(4) = 0$, and $|2\rangle$ and $|3\rangle$ are odd, $d(2) = d(3) = 1$; or (ii) $|1\rangle$ and $|4\rangle$ are odd, $d(1) = d(4) = 1$, and $|2\rangle$ and $|3\rangle$ are even, $d(2) = d(3) = 0$. We take a simple root system $\{\alpha_1, \alpha_2\}$ that is purely fermionic. Then the Chevalley generators $\{e_i^V, f_i^V, h_i^V\}_{i=1,2}$ (satisfying (A.1)) in the representation V are given by

$$\begin{aligned}
e_1^V &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = E_{12} + E_{34}, & f_1^V &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = -E_{21} + E_{43} \\
e_2^V &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = E_{13} + E_{24}, & f_2^V &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = -E_{31} + E_{42} \\
h_1^V &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = -E_{11} - E_{22} + E_{33} + E_{44} \\
h_2^V &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = -E_{11} + E_{22} - E_{33} + E_{44}.
\end{aligned} \tag{3.59}$$

Here the E_{ij} 's are matrices with the only non-zero elements in the i th row and j th column, $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$. The states $|i\rangle$ can be labeled by the eigenvalues of the h_i^V 's

$$|1\rangle = (-1, -1), \quad |2\rangle = (-1, 1), \quad |3\rangle = (1, -1), \quad |4\rangle = (1, 1). \quad (3.60)$$

This representation can be affinized [48], meaning that we can define additional generators $\{e_0^V, f_0^V, h_0^V\}$ in V such that $\{e_i^V, f_i^V, h_i^V\}_{i=1,2} \cup \{xe_0^V, x^{-1}f_0^V, h_0^V\}$ satisfy the affine relations for $osp(2|2)^{(1)}$ on the loop space $V \otimes C[x, x^{-1}]$ (with zero center). The even simple root of the affine extension is $\alpha_0 = -(\alpha_1 + \alpha_2)$ and the even generators are

$$e_0^V = \frac{1}{\sqrt{2}}\{f_1^V, f_2^V\} = -\sqrt{2}E_{41}, \quad f_0^V = \frac{1}{\sqrt{2}}\{e_1^V, e_2^V\} = \sqrt{2}E_{14} \quad (3.61a)$$

$$h_0^V = -(h_1^V + h_2^V) = 2E_{11} - 2E_{44}, \quad (3.61b)$$

where $\{\cdot, \cdot\}$ denotes the anticommutator. With an abuse of notation, the affinized representation on the loop space $V \otimes C[x, x^{-1}]$ will also be denoted by V .

Considering next the quantum algebras, the $osp(2|2)$ fundamental vector representation V is undeformed as a representation of $U_q[osp(2|2)]$. The generators of $U_q[osp(2|2)]$ satisfying the relations (A.9) have the same matrix representations as above. As in the classical case, V is also affinizable at the quantum level. However for the quantum affine algebra, the even non-Cartan generators are not given by (3.61a) but are deformed to

$$e_{0q}^V = -\sqrt{[2]_q}E_{41}, \quad f_{0q}^V = \sqrt{[2]_q}E_{14}, \quad (3.62)$$

where we have used the notation

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}. \quad (3.63)$$

The set $\{e_i^V, f_i^V, h_i^V\}_{i=1,2} \cup \{xe_{0q}^V, x^{-1}f_{0q}^V, h_0^V\}$, forms a representation of $U_q[osp(2|2)^{(1)}]$ on the loop space $V \otimes C[x, x^{-1}]$. In the limit $q \rightarrow \pm 1$, we recover $osp(2|2)^{(1)}$ (or $osp(2|2)$ from $U_q[osp(2|2)]$). Note that for $q \rightarrow -1$ we have

$$e_{0q}^V \rightarrow ie_0^V, \quad f_{0q}^V \rightarrow if_0^V, \quad [h_0]_q \rightarrow -h_0, \quad (3.64)$$

and the relation (A.9c) becomes

$$[e_{0q}^V, f_{0q}^V] = [h_0]_q \xrightarrow{q \rightarrow -1} [ie_0^V, if_0^V] = -h_0, \quad (3.65)$$

thus recovering the classical affine relation. Henceforth we drop the q subscript from (3.62) since it will be either clear from the context or explicitly stated whether we are dealing with the quantum or classical case.

The $U_q[osp(2|2)]$ representation (3.59) is referred to as being “typical” and type 1 or 2 grade star. An irreducible representation V is said to be typical if any reducible representation W that contains V can always be written as the direct sum $W = \tilde{W} \oplus V$, for some other representation \tilde{W} . A representation that is not typical is said to be “atypical”. An atypical representation will occur in the decomposition of $V \otimes V$ below. The label type 1 or type 2 grade star indicates how the representation matrices behave under hermitian conjugation [46]. A type 1 grade star representation has its generators satisfying

$$\begin{aligned} (e_i^V)_{\alpha\beta} &= (-1)^{d(f_i^V)d(\alpha)} \overline{(f_i^V)_{\beta\alpha}} \\ (f_i^V)_{\alpha\beta} &= (-1)^{d(e_i^V)d(\beta)} \overline{(e_i^V)_{\beta\alpha}} \\ (h_i^V)_{\alpha\beta} &= \overline{(h_i^V)_{\beta\alpha}}, \end{aligned} \quad (3.66)$$

where the overbar means complex conjugation and the subscripts α, β denote matrix elements. For a type 2 grade star representation we have

$$\begin{aligned} (e_i^V)_{\alpha\beta} &= (-1)^{d(f_i^V)d(\beta)} \overline{(f_i^V)_{\beta\alpha}} \\ (f_i^V)_{\alpha\beta} &= (-1)^{d(e_i^V)d(\alpha)} \overline{(e_i^V)_{\beta\alpha}} \end{aligned}$$

$$(h_i^V)_{\alpha\beta} = \overline{(h_i^V)_{\beta\alpha}}. \quad (3.67)$$

If the parities are chosen to be $d(1) = d(4) = 0$ and $d(2) = d(3) = 0$, then (3.59) is of type 2 grade star, whereas for the second choice $d(1) = d(4) = 1$ and $d(2) = d(3) = 0$, (3.59) becomes type 1 grade star.

We now want to construct the fields which will asymptotically create the states/particles for the fundamental vector representation. The topological charges of these fields must agree with (3.60). Recall that the independent topological charges are $(T^1, T^2) = -(h_1, h_2)$. Furthermore, under the action of the charges the fields must transform in a manner consistent with (3.59), (3.61b) and (3.62). This ensures that the states will form a representation of the algebra (A.9). Specifically the charges should take the form

$$\overline{Q}^i \propto e_i^V, \quad Q^i \propto f_i^V \quad (3.68a)$$

$$T^i = -h_i, \quad (3.68b)$$

when acting on the fields.

In general, the fields will consist of vertex operators multiplied by some functions $\{f_i, \overline{f}_i\}$ of the fermions

$$\Psi_i = \exp\left(\frac{i}{\beta}\omega_i \cdot \vec{\phi}\right) f_i(\eta, \partial\xi), \quad \overline{\Psi}_i = \exp\left(-\frac{i}{\beta}\omega_i \cdot \vec{\overline{\phi}}\right) \overline{f}_i(\overline{\eta}, \partial\overline{\xi}), \quad (3.69)$$

where the ω_i 's are the weights of the representation. Both sets $\{\Psi_i\}$ and $\{\overline{\Psi}_i\}$ have the same topological charges. Any set of fields differing from (3.69) by some local operators, such as

$$\chi = e^{i\alpha \cdot \vec{\Phi}}, \quad (3.70)$$

will have the same topological charges. This means that (3.68) should be viewed as modulo any local fields. Thus knowing the topological charges, we can look for two sets of fields, which will generate all other “topologically equivalent” families. We find the following fundamental fields giving the correct charges

$$\begin{aligned} \Psi_1(x, t) &= \exp\left(\frac{i}{\beta}\omega_1 \cdot \vec{\phi}(x, t)\right) \eta(z) \\ \Psi_2(x, t) &= \exp\left(\frac{i}{\beta}\omega_2 \cdot \vec{\phi}(x, t)\right) \\ \Psi_3(x, t) &= \exp\left(\frac{i}{\beta}\omega_3 \cdot \vec{\phi}(x, t)\right) \\ \Psi_4(x, t) &= \exp\left(\frac{i}{\beta}\omega_4 \cdot \vec{\phi}(x, t)\right) \partial_z \xi(z), \end{aligned} \quad (3.71)$$

and

$$\begin{aligned} \overline{\Psi}_1(x, t) &= \exp\left(\frac{i}{\beta}\omega_1 \cdot \vec{\overline{\phi}}(x, t)\right) \overline{\eta}(\overline{z}) \\ \overline{\Psi}_2(x, t) &= \exp\left(\frac{i}{\beta}\omega_2 \cdot \vec{\overline{\phi}}(x, t)\right) \\ \overline{\Psi}_3(x, t) &= \exp\left(\frac{i}{\beta}\omega_3 \cdot \vec{\overline{\phi}}(x, t)\right) \\ \overline{\Psi}_4(x, t) &= \exp\left(\frac{i}{\beta}\omega_4 \cdot \vec{\overline{\phi}}(x, t)\right) \partial_{\overline{z}} \overline{\xi}(\overline{z}). \end{aligned} \quad (3.72)$$

The weights for the fundamental representation are given by

$$\begin{aligned} \omega_1 &= (0, -i) = -\frac{1}{2}(\alpha_1 + \alpha_2) \\ \omega_2 &= (1, 0) = \frac{1}{2}(\alpha_1 - \alpha_2) \end{aligned}$$

$$\begin{aligned}\omega_3 &= (-1, 0) = -\frac{1}{2}(\alpha_1 - \alpha_2) \\ \omega_4 &= (0, i) = \frac{1}{2}(\alpha_1 + \alpha_2).\end{aligned}\tag{3.73}$$

The topological charges can be found using (3.50) and agree with (3.60). Therefore the fields (3.71) and (3.72) can be taken to produce the states $|i\rangle_{i=1,\dots,4}$ asymptotically ($t \rightarrow \pm\infty$)

$$|i, \theta\rangle = \Psi_i|0\rangle, \quad |i, \theta\rangle = \bar{\Psi}_i|0\rangle,\tag{3.74}$$

where θ is the rapidity parameterizing the energy-momentum (m is the mass)

$$E = m \cosh \theta, \quad P = m \sinh \theta.\tag{3.75}$$

Since the theory (3.27) is in a massive phase, the states (3.74) are massive particle states with the dispersion relation $E^2 = P^2 + m^2$. One needs to check that the fields transform according to (3.68). That the topological charges are correct confirms (3.68b). Using the same procedure as in the previous section, we obtain for the non-local charges acting on the fields

$$\begin{aligned}\bar{Q}^i(\Psi_j) &= \rho_i^{(2)} g \hat{\gamma} : \sum_k (e_i^V)_{kj} \Psi_k \chi_i : \equiv g \sum_k (e_i^V)_{kj} \hat{\Psi}_{ki} \\ Q^i(\Psi_j) &= \rho_i^{(2)} g \hat{\gamma} : \sum_k (f_i^V)_{kj} \bar{\Psi}_k \chi_i : \equiv g \sum_k (f_i^V)_{kj} \hat{\bar{\Psi}}_{ki},\end{aligned}\tag{3.76}$$

where the χ_i 's are local fields

$$\chi_i = \exp \left(-i \frac{\hat{\beta}}{\hat{\gamma}} \alpha_i \cdot \vec{\Phi} \right),\tag{3.77}$$

and $\{e_i^V, f_i^V\}_{i=0,1,2}$ are given by (3.59) and (3.61). From (3.76) we see that (3.68) holds. Note that the deformed factors $\sqrt{[2]_q}$ are not obtained for the action of Q^0 and \bar{Q}^0 . This is not problematic since overall factors can be adjusted by redefining the fields $\{\hat{\Psi}_{ki}, \hat{\bar{\Psi}}_{ki}\}$ without changing their particle creation properties. Lastly, all the fields have a non-trivial Lorentz spin. The equations (3.76) should have a consistent spin structure, namely if $(e_i^V)_{kj}$ or $(f_i^V)_{kj}$ is non-zero, then the spins must satisfy

$$s(\bar{Q}^i) + s(\Psi_j) = s(\hat{\Psi}_{ki}), \quad s(Q^i) + s(\bar{\Psi}_j) = s(\hat{\bar{\Psi}}_{ki}).\tag{3.78}$$

These relations are easily verified. The Lorentz spins of the charges (3.39) are encoded by the on-shell operators $e^{s\theta}$. Taking into account all the above results, the charges acting in the fundamental representation, with states generated asymptotically by the fields (3.71) and (3.72), are given by

$$\begin{aligned}\bar{Q}^i &= \rho_i^{(2)} e^{\alpha_i^2 \theta / 2 \hat{\gamma}} e_i^V q^{h_i^V / 2} \\ Q^i &= \rho_i^{(2)} e^{-\alpha_i^2 \theta / 2 \hat{\gamma}} f_i^V q^{h_i^V / 2} \\ T^i &= -h_i^V,\end{aligned}\tag{3.79}$$

where $\{e_i^V, f_i^V, h_i^V\}_{i=0,1,2}$ are the $U_q[osp(2|2)^{(1)}]$ matrices (3.59), (3.61b) and (3.62). The Lorentz factors $\exp(\pm \alpha_i^2 \theta / 2 \hat{\gamma})$ play the role of the spectral parameter $x^{\pm 1}$. Since $\alpha_1^2 = \alpha_2^2 = 0$, the above representation corresponds to the homogeneous gradation (not to be confused with the even/odd parity gradation). For a non-zero background charge \mathcal{Q} (3.31), the θ dependence for \bar{Q}^i (Q^i), $i = 0, 1, 2$, becomes $\exp(-2\theta/3\hat{\gamma})$ ($\exp(+2\theta/3\hat{\gamma})$). This corresponds to the principal gradation. The two gradations are related by an inner automorphism. Given an element $a(\theta) \in U_q[osp(2|2)^{(1)}]$ in the homogeneous gradation, denoted $a_{\text{homo.}}(\theta)$, the associated element in the principal gradation, $a_{\text{princ.}}(\theta)$, can be obtained from the transformation

$$a_{\text{princ.}}(\theta) = \sigma a_{\text{homo.}}(\theta) \sigma^{-1},\tag{3.80}$$

where $\sigma = \exp(-h_0^V \theta / 3\hat{\gamma})$.

The action of the charges on multi-particle states is given by the comultiplication. A non-trivial comultiplication arises due to the non-locality of the charges and fundamental fields. As in (3.43), the non-locality leads to non-trivial braiding between the currents and the fields. We find the braiding relations

$$\begin{aligned} J^i(x, t) \bar{\Psi}_j(y, t) &= (-1)^{d_i d(j)} q^{-T^i(\bar{\Psi}_j)} \bar{\Psi}_j(y, t) J^i(x, t) \\ \bar{J}^i(x, t) \Psi_j(y, t) &= (-1)^{d_i d(j)} q^{-T^i(\Psi_j)} \Psi_j(y, t) \bar{J}^i(x, t) \\ H^i(x, t) \bar{\Psi}_j(y, t) &= (-1)^{d_i d(j)} q^{-T^i(\bar{\Psi}_j)} \bar{\Psi}_j(y, t) H^i(x, t) \\ \bar{H}^i(x, t) \Psi_j(y, t) &= (-1)^{d_i d(j)} q^{-T^i(\Psi_j)} \Psi_j(y, t) \bar{H}^i(x, t). \end{aligned} \quad (3.81)$$

Here $d(i)$ gives the parity of the fields Ψ_i and $\bar{\Psi}_i$, and hence of the state $|i\rangle$: $d(i) = d(\Psi_i) = d(\bar{\Psi}_i)$. (Since $\Psi_{1,4}$ (or $\bar{\Psi}_{1,4}$) contain a single fermion, we could assign the parities $d(1) = d(4) = 1$ and $d(2) = d(3) = 0$. However we will not make any specific choice and treat both cases (i) and (ii) above consecutively.) It follows from (3.81) that the comultiplication is

$$\begin{aligned} \Delta(\bar{Q}^i) &= \bar{Q}^i \otimes 1 + q^{h_i} \otimes \bar{Q}^i \\ \Delta(Q^i) &= Q^i \otimes 1 + q^{h_i} \otimes Q^i \\ \Delta(h_i) &= h_i \otimes 1 + 1 \otimes h_i. \end{aligned} \quad (3.82)$$

The expression for $\Delta(h_i)$ is a consequence of the topological charge T^i being additive. The comultiplication should be understood as being implicitly graded, via the product definition

$$(A \otimes B)(a \otimes b) = (-1)^{d(B)d(a)} Aa \otimes Bb, \quad (3.83)$$

for charges A, B and states/fields a, b . This definition will be taken to hold for all quantities with definite parities and where the products/actions Aa and Ba makes sense. Since the comultiplication preserves the algebra, it provides a representation of (3.56) on multi-particle states. By defining a counit and an antipode, the quantum algebra $U_q[\mathfrak{osp}(2|2)^{(1)}]$ can be given the structure of a Hopf algebra [49–51]. We will only need the antipode (see section 4d), along with the comultiplication, to derive the S -matrix and hence avoid any further discussion of this additional algebraic structure.

4. The S -matrix

4a. Defining the S -matrix

Given a theory with quantum affine symmetry $U_q(\hat{g})$ [52], the one-particle states can be arranged into sets, or multiplets, transforming according to the irreducible representations of $U_q(\hat{g})$. Each typical representation can be uniquely labeled by the eigenvalues of the various Casimir operators in the theory. Specifically, all particles in a given multiplet have the same mass. Quantum group (or Yangian) symmetry ensures that there exists an infinite set of conserved higher-spin charges in involution, implying quantum integrability of the system. Any scattering process is strongly constrained by these conservation laws. A general multi-particle scattering event factorizes into a series of two-particle processes, with the set of incoming momenta and masses being equal to the set of outgoing momenta and masses. This means that the rapidities cannot change and that only particles belonging to the same representation can transform into one another.

Consider two multiplets, $\{|i\rangle_\alpha\}_{i=1,\dots,m}$ and $\{|j\rangle_\beta\}_{j=1,\dots,n}$, forming a basis for two representations of $U_q(\hat{g})$, V_α and V_β respectively. An asymptotic incoming two-particle state, with rapidities θ_1 and θ_2 , can be represented as

$$|i, \theta_1\rangle_\alpha \otimes |j, \theta_2\rangle_\beta, \quad (4.1)$$

where we take $\theta_1 > \theta_2$ and the notation implies a spacial ordering, i.e., the particles appear in space arranged from left to right with decreasing rapidities. The state (4.1) will scatter into some asymptotic outgoing state which, due to the above restrictions, will be of the form

$$|k, \theta_2\rangle_\beta \otimes |l, \theta_1\rangle_\alpha. \quad (4.2)$$

The outgoing states will necessarily be spacially arranged from left to right with increasing rapidities. The two-particle S -matrix is an operator, depending on the rapidities and the deformation parameter q (or $\hat{\beta}$), which relates the incoming state (4.1) to the outgoing state (4.2)

$$S^{\alpha\beta}(\theta_1, \theta_2; q) : V_\alpha \otimes V_\beta \longrightarrow V_\beta \otimes V_\alpha \quad (4.3)$$

$$|i, \theta_1\rangle_\alpha \otimes |j, \theta_2\rangle_\beta = \sum_{k,l} [S^{\alpha\beta}(\theta_1, \theta_2; q)]_{ij}^{kl} |k, \theta_2\rangle_\beta \otimes |l, \theta_1\rangle_\alpha. \quad (4.4)$$

The matrix element $[S^{\alpha\beta}(\theta_1, \theta_2; q)]_{ij}^{kl}$ gives the two particle scattering amplitude for the process

$$|i, \theta_1\rangle_\alpha \otimes |j, \theta_2\rangle_\beta \longrightarrow |k, \theta_2\rangle_\beta \otimes |l, \theta_1\rangle_\alpha. \quad (4.5)$$

Lorentz invariance requires that $S^{\alpha\beta}(\theta_1, \theta_2; q)$ only depends on the combination $\theta \equiv \theta_1 - \theta_2$

$$S^{\alpha\beta}(\theta_1, \theta_2; q) = S^{\alpha\beta}(\theta; q). \quad (4.6)$$

Since we are interested in the fundamental vector representation, $V_\alpha = V_\beta = V$, and we will drop the α, β indices.

The S -matrix has to satisfy certain constraints. Factorized scattering requires that $S(\theta; q)$ be a solution of the Yang-Baxter equation [1–3, 7, 53]. This fixes $S(\theta; q)$ up to an overall scalar constant, which can be found by imposing the crossing and unitarity conditions [7, 53]. The resulting S -matrix, known as the minimal S -matrix, is ambiguous only up to CDD factors. Lastly, applying the bootstrap program fixes the CDD factors thus giving the complete S -matrix [53–55]. We will calculate the beta dependent minimal S -matrix for the fundamental vector representation, leaving the bootstrap analysis for a future problem. The marginal S -matrix will then be obtained in the limit $\hat{\beta} \rightarrow 1$.

4b. The Yang-Baxter equation and R -matrices

The quantum charges (3.56) generate a symmetry of the theory. Therefore the action of the S -matrix must commute with the action, or comultiplication, of the charges

$$[S, \Delta(\bar{Q}^i)] = [S, \Delta(Q^i)] = [S, \Delta(h_i)] = 0. \quad (4.7)$$

For the fundamental representation (3.79), these commutation relations take the explicit form

$$S(\theta; q) \left(x_{1i} e_i^V q^{h_i^V/2} \otimes 1 + q^{h_i^V} \otimes x_{2i} e_i^V q^{h_i^V/2} \right) = \left(x_{2i} e_i^V q^{h_i^V/2} \otimes 1 + q^{h_i^V} \otimes x_{1i} e_i^V q^{h_i^V/2} \right) S(\theta; q) \quad (4.8a)$$

$$S(\theta; q) \left(x_{1i}^{-1} f_i^V q^{h_i^V/2} \otimes 1 + q^{h_i^V} \otimes x_{2i}^{-1} f_i^V q^{h_i^V/2} \right) = \left(x_{2i}^{-1} f_i^V q^{h_i^V/2} \otimes 1 + q^{h_i^V} \otimes x_{1i}^{-1} f_i^V q^{h_i^V/2} \right) S(\theta; q) \quad (4.8b)$$

$$S(\theta; q) (h_i^V \otimes 1 + 1 \otimes h_i^V) = (h_i^V \otimes 1 + 1 \otimes h_i^V) S(\theta; q), \quad (4.8c)$$

where

$$x_{ji} = e^{\alpha_i^2 \theta_j / 2\hat{\gamma}}, \quad i = 0, 1, 2; \quad j = 1, 2. \quad (4.9)$$

For the fermionic roots we simply have $x_{j1} = x_{j2} = 1$. Since $h_0^V = -(h_1^V + h_2^V)$, the equation involving $\Delta(h_0^V)$ need not be considered independently. Multiplying both sides of (4.8) by $q^{-h_i^V/2} \otimes q^{-h_i^V/2}$ from the right, and making use of

$$S(\theta; q) \left(q^{-h_i^V/2} \otimes q^{-h_i^V/2} \right) = \left(q^{-h_i^V/2} \otimes q^{-h_i^V/2} \right) S(\theta; q), \quad (4.10)$$

which follows from (4.8c), the expressions (4.8) can be rewritten as

$$[S(x; q), \Delta(e_{1,2}^V)] = 0 \quad (4.11a)$$

$$[S(x; q), \Delta(e_0^V)] = 0 \quad (4.11b)$$

$$[S(x; q), \Delta(f_{1,2}^V)] = 0 \quad (4.11c)$$

$$[S(x; q), \Delta(f_0^V)] = 0 \quad (4.11d)$$

$$[S(x; q), \Delta(h_{1,2}^V)] = 0, \quad (4.11e)$$

where $\Delta(h_{1,2}^V)$ is as in (3.82) and

$$\begin{aligned} \Delta(e_{1,2}^V) &= e_{1,2}^V \otimes q^{-h_{1,2}^V/2} + q^{h_{1,2}^V/2} \otimes e_{1,2}^V \\ \Delta(e_0^V) &= x_{10} e_0^V \otimes q^{-h_0^V/2} + q^{h_0^V/2} \otimes x_{20} e_0^V \\ \Delta(f_{1,2}^V) &= f_{1,2}^V \otimes q^{-h_{1,2}^V/2} + q^{h_{1,2}^V/2} \otimes f_{1,2}^V \\ \Delta(f_0^V) &= x_{10}^{-1} f_0^V \otimes q^{-h_0^V/2} + q^{h_0^V/2} \otimes x_{20}^{-1} f_0^V. \end{aligned} \quad (4.12)$$

The comultiplication (4.12) is the standard comultiplication for the Chevalley generators of $U_q[osp(2|2)^{(1)}]$. For the affine generators, e_0 and f_0 , the comultiplication corresponds to the representation $e_0 = x_{i0} e_0^V$ and $f_0 = x_{i0}^{-1} f_0^V$ over the loop algebra. We have also defined the spectral parameter

$$x = \frac{x_{10}}{x_{20}} = e^{-2(\theta_1 - \theta_2)/\gamma} = e^{-2\theta/\gamma}, \quad (4.13)$$

and indicated the dependence of S on θ implicitly via x .

Equation (4.11) is precisely one of the defining relations for the $U_q[osp(2|2)^{(1)}]$ R -matrix $\check{R}(x; q)$ in the homogeneous gradation. More generally, for affinizable representations V_α and V_β , $\check{R}^{\alpha\beta}(x)$ acts as $S(x)$

$$\check{R}(x; q) : V_\alpha \otimes V_\beta \longrightarrow V_\beta \otimes V_\alpha, \quad (4.14)$$

and satisfies the intertwining property [51]

$$\begin{aligned} \check{R}^{\alpha\beta}(x, q) \Delta^{\alpha\beta}(a) &= \Delta^{\alpha\beta}(a) \check{R}^{\alpha\beta}(x, q), \quad \forall a \in U_q[osp(2|2)] \\ \check{R}^{\alpha\beta}(x; q) \left(x e_0^\alpha \otimes q^{-h_0^\beta/2} + q^{h_0^\alpha/2} \otimes e_0^\beta \right) &= \left(e_0^\beta \otimes q^{-h_0^\alpha/2} + q^{h_0^\beta/2} \otimes x e_i^\alpha \right) \check{R}^{\alpha\beta}(x; q) \\ \check{R}^{\alpha\beta}(x; q) \left(x^{-1} f_0^\alpha \otimes q^{-h_0^\beta/2} + q^{h_0^\alpha/2} \otimes f_0^\beta \right) &= \left(f_0^\beta \otimes q^{-h_0^\alpha/2} + q^{h_0^\beta/2} \otimes x^{-1} f_i^\alpha \right) \check{R}^{\alpha\beta}(x; q), \end{aligned} \quad (4.15)$$

where all quantities are evaluated in the appropriate representation as indicated by the α, β indices. One can also view $\check{R}^{\alpha\beta}(x)$ as the spectral parameter dependent R -matrix of $U_q[osp(2|2)]$. This equivalence follows from the fact that for affinizable representations, both the $U_q[osp(2|2)^{(1)}]$ R -matrix (in the homogeneous gradation) and the spectral parameter dependent R -matrix of $U_q[osp(2|2)]$ satisfy the same defining relation (4.15). Another R -matrix $R(x; q)$, commonly referred to as the universal R -matrix, can be obtained from $\check{R}(x; q)$ by applying a permutation operation. Define the graded permutation operator P satisfying

$$P(|u\rangle \otimes |v\rangle) = (-1)^{d(u)d(v)} |v\rangle \otimes |u\rangle, \quad (4.16)$$

for any states $|u\rangle$ and $|v\rangle$ in V with definite parity. The universal R -matrix is related to $\check{R}(x; q)$ by

$$R(x; q) = P \check{R}(x; q). \quad (4.17)$$

The universal R -matrix is known for various quantum affine superalgebras (see [51] and references therein).

For bosonic quantum groups, Jimbo [4] showed that a solution to (4.15) is unique up to an overall scalar function, and that any solution automatically satisfies the Yang-Baxter equation. These results were extended to quantum supergroups by Bracken *et al.* [50]. These authors showed that a non-trivial solution of (4.11a) and (4.11b) must be even and is unique up to scalar factors. Furthermore, this solution satisfies (i) equations (4.11c)–(4.11e) and (ii) the Yang-Baxter equation. The uniqueness of the solution implies that $S(x; q)$ takes the form

$$S(x; q) = v(x; q) \check{R}(x; q), \quad (4.18)$$

where the scalar function $v(x; q)$ is fixed by imposing the unitarity and crossing constraints (see section 4e) and by applying the bootstrap principle. Thus for a theory with quantum group symmetry, the Yang-Baxter equation need not be independently solved. Rather factorization is a consequence of the intertwining property (4.15) of the R -matrix.

The Yang-Baxter equations satisfied by the two R -matrices are

$$R_{12}(x; q)R_{13}(xy; q)R_{23}(y; q) = R_{23}(y; q)R_{13}(xy; q)R_{12}(x; q) \quad (4.19)$$

$$\check{R}_{23}(x; q)\check{R}_{12}(xy; q)\check{R}_{23}(y; q) = \check{R}_{12}(y; q)\check{R}_{23}(xy; q)\check{R}_{12}(x; q). \quad (4.20)$$

Both equations act on $V \otimes V \otimes V$ and for a solution $R = \sum_i a_i \otimes b_i$, the notation R_{ij} means

$$R_{12} = \sum_i a_i \otimes b_i \otimes 1, \quad R_{13} = \sum_i a_i \otimes 1 \otimes b_i, \quad R_{23} = \sum_i 1 \otimes a_i \otimes b_i, \quad (4.21)$$

and similarly for \check{R}_{ij} . The Yang-Baxter equation for $R(x; q)$ is implicitly graded due to the multiplication rule (3.83). In components (4.19) takes the form (summing over repeated indices)

$$\begin{aligned} R(x; q)^{b_1 b_2}_{c_1 c_2} R(xy; q)^{c_1 b_3}_{a_1 c_3} R(y; q)^{c_2 c_3}_{a_2 a_3} (-1)^{d(b_1)d(b_2)+d(c_1)d(b_3)+d(c_2)d(c_3)} = \\ R(y; q)^{b_2 b_3}_{c_2 c_3} R(xy; q)^{b_1 c_3}_{c_1 a_3} R(x; q)^{c_1 c_2}_{a_1 a_2} (-1)^{d(b_2)d(b_3)+d(b_1)d(c_3)+d(c_1)d(c_2)}. \end{aligned} \quad (4.22)$$

The R -matrix $\check{R}(x; q)^{b_1 b_2}_{a_1 a_2} \equiv R(x; q)^{b_1 b_2}_{a_1 a_2} (-1)^{d(b_1)d(b_2)}$ satisfies the ordinary Yang-Baxter equation. The component form of (4.20) can be obtained from (4.22) by setting

$$\check{R}(x; q)^{b_1 b_2}_{a_1 a_2} = R(x; q)^{b_2 b_1}_{a_1 a_2} (-1)^{d(b_1)d(b_2)}, \quad (4.23)$$

which does not contain any parity factors. Of course the S -matrix also satisfies (4.20).

4c. The $U_q[\mathfrak{osp}(2|2)^{(1)}]$ R -matrix $\check{R}(x; q)$

A solution to (4.11)/(4.15) has been previously computed in [56, 57] and [58]. In [56] $\check{R}(x; q)$ was constructed for a set of four-dimensional typical representations characterized by a continuous parameter b . This set includes the fundamental vector representation. We will re-derive $\check{R}(x; q)$ using the method of [57] and including some of the details omitted in these references. Our conventions are also slightly different. We hope a more explicit construction of $\check{R}(x; q)$, showing the various steps involved, will be useful.

As discussed above, it is sufficient to solve the following reduced set of equations

$$[\check{R}(x; q), \Delta(e_{1,2}^V)] = 0$$

$$\check{R}(x; q) \left(x e_0^V \otimes q^{-h_0^V/2} + q^{h_0/2} \otimes e_0^V \right) = \left(e_0^V \otimes q^{-h_0^V/2} + q^{h_0/2} \otimes x e_0^V \right) \check{R}(x; q). \quad (4.24)$$

However it is equally as manageable, and perhaps more illustrative, to work with the set

$$[\check{R}(x; q), \Delta(a)] = 0, \quad \forall a \in U_q[\mathfrak{osp}(2|2)] \quad (4.25a)$$

$$\check{R}(x; q) \left(x e_0^V \otimes q^{-h_0^V/2} + q^{h_0/2} \otimes e_0^V \right) = \left(e_0^V \otimes q^{-h_0^V/2} + q^{h_0/2} \otimes x e_0^V \right) \check{R}(x; q), \quad (4.25b)$$

which obviously includes (4.24). Thus we will seek a solution to (4.25).

Consider for a moment the general case (4.15), with (4.25) being evaluated in $V_\alpha \otimes V_\beta$ for affinizable representations V_α and V_β . Suppose $V_\alpha \otimes V_\beta$ has the multiplicity-free tensor product decomposition

$$V_\alpha \otimes V_\beta = \bigoplus_\mu V_\mu, \quad (4.26)$$

into $U_q[\mathfrak{osp}(2|2)]$ invariant spaces V_μ . Since $\check{R}(x; q)$ commutes with the $U_q[\mathfrak{osp}(2|2)]$ comultiplication, a solution of (4.25a) can be written as

$$\check{R}(x; q) = \sum_\mu \rho_\mu(x; q) \mathcal{P}_\mu^{\alpha\beta}(q), \quad (4.27)$$

where $\mathcal{P}_\mu^{\alpha\beta}(q)$ are projectors onto V_μ and $\rho_\mu(x; q)$ are arbitrary functions. If the decomposition (4.26) is completely reducible, then (4.27) is the most general solution of (4.25a). If instead the decomposition is not fully reducible, then (4.27) need not be the most general solution, though it certainly is one solution.

Returning to the fundamental representation, we look for a decomposition of $V \otimes V$ into $U_q[osp(2|2)]$ invariant spaces. In the classical case, the tensor product is not completely reducible. One finds a decomposition into two eight-dimensional invariant spaces [43]

$$V \otimes V = W \oplus \tilde{W}, \quad (4.28)$$

where W is an irreducible $osp(2|2)$ representation spanned by

$$\begin{aligned} |\chi_1^\pm\rangle &= |1\rangle \otimes |1\rangle \\ |\chi_2^\pm\rangle &= \frac{1}{\sqrt{2}}(|1\rangle \otimes |2\rangle \pm |2\rangle \otimes |1\rangle) \\ |\chi_3^\pm\rangle &= \frac{1}{\sqrt{2}}(|1\rangle \otimes |3\rangle \pm |3\rangle \otimes |1\rangle) \\ |\chi_4^\pm\rangle &= \frac{1}{\sqrt{2}}(|2\rangle \otimes |3\rangle - |3\rangle \otimes |2\rangle) \\ |\chi_5^\pm\rangle &= \frac{1}{\sqrt{2}}(|1\rangle \otimes |4\rangle + |4\rangle \otimes |1\rangle) \\ |\chi_6^\pm\rangle &= \frac{1}{\sqrt{2}}(|2\rangle \otimes |4\rangle \pm |4\rangle \otimes |2\rangle) \\ |\chi_7^\pm\rangle &= \frac{1}{\sqrt{2}}(|3\rangle \otimes |4\rangle \pm |4\rangle \otimes |3\rangle) \\ |\chi_8^\pm\rangle &= |4\rangle \otimes |4\rangle, \end{aligned} \quad (4.29)$$

and \tilde{W} is an atypical representation spanned by

$$\begin{aligned} |\chi_9^\pm\rangle &= |2\rangle \otimes |2\rangle \\ |\chi_{10}^\pm\rangle &= \frac{1}{\sqrt{2}}(|1\rangle \otimes |2\rangle \mp |2\rangle \otimes |1\rangle) \\ |\chi_{11}^\pm\rangle &= \frac{1}{\sqrt{2}}(|1\rangle \otimes |3\rangle \mp |3\rangle \otimes |1\rangle) \\ |\chi_{12}^\pm\rangle &= \frac{1}{2}(|1\rangle \otimes |4\rangle - |4\rangle \otimes |1\rangle \mp |2\rangle \otimes |3\rangle \mp |3\rangle \otimes |2\rangle) \\ |\chi_{13}^\pm\rangle &= \frac{1}{2}(|1\rangle \otimes |4\rangle - |4\rangle \otimes |1\rangle \pm |2\rangle \otimes |3\rangle \pm |3\rangle \otimes |2\rangle) \\ |\chi_{14}^\pm\rangle &= \frac{1}{\sqrt{2}}(|2\rangle \otimes |4\rangle \mp |4\rangle \otimes |2\rangle) \\ |\chi_{15}^\pm\rangle &= \frac{1}{\sqrt{2}}(|3\rangle \otimes |4\rangle \mp |4\rangle \otimes |3\rangle) \\ |\chi_{16}^\pm\rangle &= |3\rangle \otimes |3\rangle. \end{aligned} \quad (4.30)$$

Here and henceforth the upper (lower) sign is to be taken if the parities are chosen to be even (odd) for $|1\rangle$, $|4\rangle$ and odd (even) for $|2\rangle$, $|3\rangle$, and the associated states will be labeled by a $+$ ($-$) superscript. The two sets of states

$$\{|\chi_9^\pm\rangle, |\chi_{10}^\pm\rangle, |\chi_{11}^\pm\rangle, |\chi_{12}^\pm\rangle\} \quad \text{and} \quad \{|\chi_{12}^\pm\rangle, |\chi_{14}^\pm\rangle, |\chi_{15}^\pm\rangle, |\chi_{16}^\pm\rangle\}, \quad (4.31)$$

form four-dimensional atypical representations, with $|\chi_{12}^\pm\rangle$ being invariant, i.e., is mapped to zero by all the generators. The state $|\chi_{13}^\pm\rangle$ is a cyclic vector for \tilde{W} . In the quantum case these states will be deformed, yet one expects that the basic structure of the decomposition will be the same, namely the tensor product is not completely reducible and has the form [56, 57]

$$V \otimes V = W_q \oplus \tilde{W}_q, \quad (4.32)$$

where W_q is an irreducible invariant space going over to W as $q \rightarrow 1$, and \tilde{W}_q is not irreducible, but contains an invariant singlet state and goes over to \tilde{W} as $q \rightarrow 1$. A mechanical way to determine a basis for, say W_q , is to start with one of the states $|\chi_i^\pm\rangle \in W$, or some deformed version of it. Acting on this state with the $U_q[osp(2|2)]$ generators will result in some new deformed states. One keeps repeating the process, acting on every new state with $\Delta(a)$ until an invariant set with the required properties is obtained. The success of this procedure depends on the choice of the initial state. Starting with the undeformed states $|1\rangle \otimes |1\rangle$, $|2\rangle \otimes |2\rangle$ and $|3\rangle \otimes |3\rangle$, almost all the basis states of W_q and \tilde{W}_q can be obtained this way. The exception is the cyclic state of \tilde{W}_q which has to be independently constructed. We find the following set spanning W_q

$$\begin{aligned}
|\psi_1^\pm\rangle &= |1\rangle \otimes |1\rangle \\
|\psi_2^\pm\rangle &= \frac{1}{\sqrt{q+q^{-1}}} \left(q^{-1/2}|1\rangle \otimes |2\rangle \pm q^{1/2}|2\rangle \otimes |1\rangle \right) \\
|\psi_3^\pm\rangle &= \frac{1}{\sqrt{q+q^{-1}}} \left(q^{-1/2}|1\rangle \otimes |3\rangle \pm q^{1/2}|3\rangle \otimes |1\rangle \right) \\
|\psi_4^\pm\rangle &= \frac{1}{\sqrt{2(q^2+q^{-2})}} \left(q^{-1}|1\rangle \otimes |4\rangle + q|4\rangle \otimes |1\rangle \pm q|2\rangle \otimes |3\rangle \mp q^{-1}|3\rangle \otimes |2\rangle \right) \\
|\psi_5^\pm\rangle &= \frac{1}{\sqrt{2(q^2+q^{-2})}} \left(q^{-1}|1\rangle \otimes |4\rangle + q|4\rangle \otimes |1\rangle \mp q^{-1}|2\rangle \otimes |3\rangle \pm q|3\rangle \otimes |2\rangle \right) \\
|\psi_6^\pm\rangle &= \frac{1}{\sqrt{q+q^{-1}}} \left(q^{-1/2}|2\rangle \otimes |4\rangle \pm q^{1/2}|4\rangle \otimes |2\rangle \right) \\
|\psi_7^\pm\rangle &= \frac{1}{\sqrt{q+q^{-1}}} \left(q^{-1/2}|3\rangle \otimes |4\rangle \pm q^{1/2}|4\rangle \otimes |3\rangle \right) \\
|\psi_8^\pm\rangle &= |4\rangle \otimes |4\rangle,
\end{aligned} \tag{4.33}$$

and for \tilde{W}_q a basis is given by

$$\begin{aligned}
|\psi_9^\pm\rangle &= |2\rangle \otimes |2\rangle \\
|\psi_{10}^\pm\rangle &= \frac{1}{\sqrt{q+q^{-1}}} \left(q^{1/2}|1\rangle \otimes |2\rangle \mp q^{-1/2}|2\rangle \otimes |1\rangle \right) \\
|\psi_{11}^\pm\rangle &= \frac{1}{\sqrt{q+q^{-1}}} \left(q^{1/2}|1\rangle \otimes |3\rangle \mp q^{-1/2}|3\rangle \otimes |1\rangle \right) \\
|\psi_{12}^\pm\rangle &= \frac{1}{2} (|1\rangle \otimes |4\rangle - |4\rangle \otimes |1\rangle \mp |2\rangle \otimes |3\rangle \mp |3\rangle \otimes |2\rangle) \\
|\psi_{13}^\pm\rangle &= \frac{1}{\sqrt{q^2+q^{-2}}} (q|1\rangle \otimes |4\rangle - q^{-1}|4\rangle \otimes |1\rangle) \\
|\psi_{14}^\pm\rangle &= \frac{1}{\sqrt{q+q^{-1}}} \left(q^{1/2}|2\rangle \otimes |4\rangle \mp q^{-1/2}|4\rangle \otimes |2\rangle \right) \\
|\psi_{15}^\pm\rangle &= \frac{1}{\sqrt{q+q^{-1}}} \left(q^{1/2}|3\rangle \otimes |4\rangle \mp q^{-1/2}|4\rangle \otimes |3\rangle \right) \\
|\psi_{16}^\pm\rangle &= |3\rangle \otimes |3\rangle.
\end{aligned} \tag{4.34}$$

One can check that:

(i)

$$\lim_{q \rightarrow 1} W_q = \lim_{q \rightarrow 1} \text{span}\{|\psi_i^\pm\rangle; i = 1, \dots, 8\} = W, \quad \lim_{q \rightarrow 1} \tilde{W}_q = \lim_{q \rightarrow 1} \text{span}\{|\psi_i^\pm\rangle; i = 9, \dots, 16\} = \tilde{W}; \tag{4.35}$$

(ii) the set (4.33) forms a typical representation of $U_q[osp(2|2)]$; (iii) the space \tilde{W}_q is composed of two atypical $U_q[osp(2|2)]$ representations spanned by

$$\{|\psi_9^\pm\rangle, |\psi_{10}^\pm\rangle, |\psi_{11}^\pm\rangle, |\psi_{12}^\pm\rangle\} \quad \text{and} \quad \{|\psi_{12}^\pm\rangle, |\psi_{14}^\pm\rangle, |\psi_{15}^\pm\rangle, |\psi_{16}^\pm\rangle\}; \tag{4.36}$$

(iv) the state $|\psi_{12}^\pm\rangle$ is a singlet state which cannot be separated from the atypical representations; and (v) $|\psi_{13}^\pm\rangle$ is a cyclic vector for \tilde{W}_q which cannot be obtained by acting on any basis state by any of the generators. The adjoint states $\langle\psi_i^\pm|$ are given by (note we do not take $q \rightarrow \bar{q}$ for the adjoint states)

$$\langle\psi_i^\pm| = (|\psi_i^\pm\rangle)^\dagger, \quad (4.37)$$

where

$$(|i\rangle \otimes |j\rangle)^\dagger = (-1)^{d(i)d(j)} \langle i| \otimes \langle j| \quad (4.38)$$

$$(|i\rangle)^\dagger = \langle i|, \quad i = 1, 2, 3, 4. \quad (4.39)$$

The parity factor in (4.38) is introduced to cancel that of (3.83), so that the norm is positive

$$(|i\rangle \otimes |j\rangle)^\dagger (|i\rangle \otimes |j\rangle) = \langle i|i\rangle \langle j|j\rangle = 1. \quad (4.40)$$

To determine $\check{R}(x; q)$ we need to know the projectors for W_q and \tilde{W}_q , denoted $\mathcal{P}_1(q)$ and $\mathcal{P}_0(q)$ respectively. If the states were all orthonormal, then $\mathcal{P}_1(q)$ and $\mathcal{P}_0(q)$ would have the usual form: $\sum_i |\psi_i^\pm\rangle \langle\psi_i^\pm|$. However the four states $\{|\psi_4^\pm\rangle, |\psi_5^\pm\rangle, |\psi_{12}^\pm\rangle, |\psi_{13}^\pm\rangle\}$ are not orthonormal. Following [57], we define the dual states $\{\langle\hat{\psi}_i^\pm|\}$ satisfying

$$\langle\hat{\psi}_i^\pm|\psi_j^\pm\rangle = \delta_{ij}. \quad (4.41)$$

Let g_{ij} be the metric

$$g_{ij} = \langle\psi_i^\pm|\psi_j^\pm\rangle, \quad (4.42)$$

then the dual states are given by

$$\langle\hat{\psi}_i^\pm| = \sum_j (g^{-1})_{ij} \langle\psi_j^\pm|. \quad (4.43)$$

For $i \notin \{4, 5, 12, 13\}$, we simply have

$$\langle\hat{\psi}_i^\pm| = \langle\psi_i^\pm|. \quad (4.44)$$

Using the dual states the projectors can be written as

$$\mathcal{P}_1(q) = \sum_{i=1}^8 |\psi_i^\pm\rangle \langle\hat{\psi}_i^\pm| \quad (4.45)$$

$$\mathcal{P}_0(q) = \sum_{i=9}^{16} |\psi_i^\pm\rangle \langle\hat{\psi}_i^\pm|. \quad (4.46)$$

It is easily shown that (4.45) and (4.46) obey

$$\mathcal{P}_1(q)^2 = \mathcal{P}_1(q), \quad \mathcal{P}_0(q)^2 = \mathcal{P}_0(q), \quad \mathcal{P}_1(q)\mathcal{P}_0(q) = \mathcal{P}_0(q)\mathcal{P}_1(q) = 0, \quad \mathcal{P}_1(q) + \mathcal{P}_0(q) = 1. \quad (4.47)$$

A particular solution for $\check{R}(x; q)$ satisfying (4.25a) is now given by

$$\check{R}_0(x; q) = \rho_1(x; q)\mathcal{P}_1(q) + \rho_0(x; q)\mathcal{P}_0(q). \quad (4.48)$$

Recalling our previous comments, since the decomposition (4.32) is not completely reducible, (4.48) is not necessarily the most general solution. In particular, the operator

$$\mathcal{P}_N(q) = |\psi_{12}^\pm\rangle \langle\hat{\psi}_{13}^\pm|, \quad (4.49)$$

mapping the cyclic vector onto the singlet state, can be added to (4.48). That this does not spoil (4.25a) follows from (iv) and (v) above, which imply

$$\Delta(a)\mathcal{P}_N(q)|\psi_i^\pm\rangle = \mathcal{P}_N(q)\Delta(a)|\psi_i^\pm\rangle = 0, \quad \forall i. \quad (4.50)$$

Some useful properties of $\mathcal{P}_N(q)$ are

$$\mathcal{P}_N(q)^2 = 0, \quad \mathcal{P}_1(q)\mathcal{P}_N(q) = \mathcal{P}_N(q)\mathcal{P}_1(q) = 0, \quad \mathcal{P}_0(q)\mathcal{P}_N(q) = \mathcal{P}_N(q)\mathcal{P}_0(q) = \mathcal{P}_N(q), \quad (4.51)$$

with the first expressing the order 2 nilpotency of $\mathcal{P}_N(q)$. Thus the most general solution of (4.25a) is

$$\check{R}(x; q) = \rho_1(x; q)\mathcal{P}_1(q) + \rho_0(x; q)\mathcal{P}_0(q) + \rho_N(x; q)\mathcal{P}_N(q), \quad (4.52)$$

where the $\rho_i(x; q)$'s are at present arbitrary functions soon to be fixed by imposing (4.25b).

Before calculating these functions, we give the explicit expressions for the projectors and $\mathcal{P}_N(q)$ in the original basis $\{|i\rangle \otimes |j\rangle; i, j = 1, \dots, 4\}$. We choose to order the basis as

$$(v_{11}, v_{22}, v_{33}, v_{44}, v_{12}, v_{21}, v_{13}, v_{31}, v_{24}, v_{42}, v_{34}, v_{43}, v_{14}, v_{23}, v_{32}, v_{41}), \quad (4.53)$$

where

$$v_{ij} \equiv |i\rangle \otimes |j\rangle. \quad (4.54)$$

Define \hat{v} as the column vector associated with (4.53), i.e., $\hat{v}_1 = v_{11}, \hat{v}_2 = v_{22}, \dots$. The basis states can be written as

$$|\psi_i^\pm\rangle = \sum_j M_{ij} \hat{v}_j, \quad (4.55)$$

where the matrix M_{ij} can be obtained from (4.33) and (4.34). The metric and the inverse metric take the form (T denotes transpose)

$$g = MM^T, \quad g^{-1} = (M^{-1})^T M^{-1}. \quad (4.56)$$

This gives the single “projectors”

$$\begin{aligned} \mathcal{P}_{kl}(q) &\equiv |\psi_k^\pm\rangle \langle \hat{\psi}_l^\pm| \\ (\mathcal{P}_{kl}(q))_{ij} &= M_{ki}(M^{-1})_{jl}, \end{aligned} \quad (4.57)$$

and hence for the required matrix components

$$(\mathcal{P}_1(x; q))_{ij} = \sum_{k=1}^8 M_{ki}(M^{-1})_{jk} \quad (4.58)$$

$$(\mathcal{P}_0(x; q))_{ij} = \sum_{k=9}^{16} M_{ki}(M^{-1})_{jk} \quad (4.59)$$

$$(\mathcal{P}_N(x; q))_{ij} = M_{12i}(M^{-1})_{j13}. \quad (4.60)$$

These expressions are easily evaluated. We find the following block diagonal form for all the operators ($\mathcal{O} = \mathcal{P}_{1,0,N}(x; q)$)

$$\mathcal{O}_{16 \times 16} = \begin{pmatrix} \mathcal{O}_{1 \times 1} & & & & & & & & & & & & & & & \\ & \mathcal{O}_{1 \times 1} & & & & & & & & & & & & & & \\ & & \mathcal{O}_{1 \times 1} & & & & & & & & & & & & & \\ & & & \mathcal{O}_{1 \times 1} & & & & & & & & & & & & \\ & & & & \mathcal{O}_{2 \times 2} & & & & & & & & & & & \\ & & & & & \mathcal{O}_{2 \times 2} & & & & & & & & & & \\ & & & & & & \mathcal{O}_{2 \times 2} & & & & & & & & & \\ & & & & & & & \mathcal{O}_{2 \times 2} & & & & & & & & \\ & & & & & & & & \mathcal{O}_{2 \times 2} & & & & & & & \\ & & & & & & & & & \mathcal{O}_{4 \times 4} & & & & & & \end{pmatrix}, \quad (4.61)$$

with the individual blocks being:

(i) One-dimensional blocks:

$$\mathcal{P}_1(q) = 1, \quad \mathcal{P}_0(q) = \mathcal{P}_N(q) = 0, \quad (4.62)$$

for $|1\rangle \otimes |1\rangle$ and $|4\rangle \otimes |4\rangle$; and

$$\mathcal{P}_0(q) = 1, \quad \mathcal{P}_1(q) = \mathcal{P}_N(q) = 0, \quad (4.63)$$

for $|2\rangle \otimes |2\rangle$ and $|3\rangle \otimes |3\rangle$.

(ii) Two-dimensional blocks:

$$\mathcal{P}_1(q) = \frac{1}{q^2 + 1} \begin{pmatrix} 1 & \pm q \\ \pm q & q^2 \end{pmatrix}, \quad \mathcal{P}_0(q) = \frac{1}{q^2 + 1} \begin{pmatrix} q^2 & \mp q \\ \mp q & 1 \end{pmatrix}, \quad \mathcal{P}_N(q) = 0, \quad (4.64)$$

for the four pairs of states

$$(|1\rangle \otimes |2\rangle, |2\rangle \otimes |1\rangle), \quad (|1\rangle \otimes |3\rangle, |3\rangle \otimes |1\rangle), \quad (|2\rangle \otimes |4\rangle, |4\rangle \otimes |2\rangle), \quad (|3\rangle \otimes |4\rangle, |4\rangle \otimes |3\rangle).$$

(ii) Four-dimensional blocks:

$$\mathcal{P}_1(q) = \frac{1}{(q^2 + 1)^2} \begin{pmatrix} 2 & \mp(q^2 - 1) & \mp(q^2 - 1) & 2q^2 \\ \pm(q^2 - 1) & 2q^2 & -q^4 - 1 & \pm(q^4 - q^2) \\ \pm(q^2 - 1) & -q^4 - 1 & 2q^2 & \pm(q^4 - q^2) \\ 2q^2 & \mp(q^4 - q^2) & \mp(q^4 - q^2) & 2q^4 \end{pmatrix}, \quad (4.65)$$

$$\mathcal{P}_0(q) = \frac{1}{(q^2 + 1)^2} \begin{pmatrix} q^4 + 2q^2 - 1 & \pm(q^2 - 1) & \pm(q^2 - 1) & -2q^2 \\ \mp(q^2 - 1) & q^4 + 1 & q^4 + 1 & \mp(q^4 - q^2) \\ \mp(q^2 - 1) & q^4 + 1 & q^4 + 1 & \mp(q^4 - q^2) \\ -2q^2 & \pm(q^4 - q^2) & \pm(q^4 - q^2) & -q^4 + 2q^2 + 1 \end{pmatrix}, \quad (4.66)$$

$$\mathcal{P}_N(q) = \frac{\sqrt{q^4 + 1}}{2(q^2 + 1)} \begin{pmatrix} 1 & \pm 1 & \pm 1 & -1 \\ \mp 1 & -1 & -1 & \pm 1 \\ \mp 1 & -1 & -1 & \pm 1 \\ -1 & \mp 1 & \mp 1 & 1 \end{pmatrix}, \quad (4.67)$$

for the basis states

$$(|1\rangle \otimes |4\rangle, |2\rangle \otimes |3\rangle, |3\rangle \otimes |2\rangle, |4\rangle \otimes |1\rangle).$$

The projectors for the two different parity assignments, denote them $\mathcal{P}_i^+(q)$ and $\mathcal{P}_i^-(q)$, are related by a similarity transformation

$$\mathcal{P}_i^+(q) = G\mathcal{P}_i^-(q)G, \quad G^2 = 1, \quad (4.68)$$

where G is the diagonal matrix

$$G = \text{diag}(1, 1, 1, 1, 1, -1, 1, -1, 1, -1, 1, -1, -1, 1, 1). \quad (4.69)$$

Obviously, the corresponding R -matrices are also related by (4.68).

Lastly, the $\rho_i(x; q)$'s are determined from (4.25b), which we rewrite below

$$\begin{aligned} & \left(\rho_1(x; q)\mathcal{P}_1(q) + \rho_0(x; q)\mathcal{P}_0(q) + \rho_N(x; q)\mathcal{P}_N(q) \right) \left(xe_0^V \otimes q^{-h_0^V/2} + q^{h_0^V/2} \otimes e_0^V \right) \\ &= \left(e_0^V \otimes q^{-h_0^V/2} + q^{h_0^V/2} \otimes xe_0^V \right) \left(\rho_1(x; q)\mathcal{P}_1(q) + \rho_0(x; q)\mathcal{P}_0(q) + \rho_N(x; q)\mathcal{P}_N(q) \right) \end{aligned} \quad (4.70)$$

Multiplying this equation by $\mathcal{P}_1(q)$ on the left and by $\mathcal{P}_N(q)$ on the right, and with the help of (4.47) and (4.51), gives

$$\begin{aligned} & \rho_1(x; q)\mathcal{P}_1(q) \left(xe_0^V \otimes q^{-h_0^V/2} + q^{h_0^V/2} \otimes e_0^V \right) \mathcal{P}_N(q) \\ &= \mathcal{P}_1(q) \left(xe_0^V \otimes q^{-h_0^V/2} + q^{h_0^V/2} \otimes e_0^V \right) \rho_0(x; q)\mathcal{P}_N(q). \end{aligned} \quad (4.71)$$

Evaluating (4.71) using the explicit expressions for the states (4.33) and (4.34), and the generators (3.61b) and (3.62), leads to the result

$$\rho_1(x; q) = \left(\frac{x - q^2}{1 - xq^2} \right) \rho_0(x; q). \quad (4.72)$$

To solve for $\rho_N(x; q)$ we multiply (4.70) on the left by $\mathcal{P}_1(q)$ and on the right by $\mathcal{P}_0(q)$ to get

$$\begin{aligned} \rho_1(x; q)\mathcal{P}_1(q) & \left(x e_0^V \otimes q^{-h_0^V/2} + q^{h_0^V/2} \otimes e_0^V \right) \mathcal{P}_0(q) \\ & = \mathcal{P}_1(q) \left(x e_0^V \otimes q^{-h_0^V/2} + q^{h_0^V/2} \otimes e_0^V \right) (\rho_0(x; q)\mathcal{P}_0(q) + \rho_0(x; q)\mathcal{P}_N(q)), \end{aligned} \quad (4.73)$$

from which it follows that

$$\rho_N(x; q) = 2q^2 \frac{q^2 - 1}{\sqrt{1 + q^4}} \frac{1 - x^2}{(x - q^2)(1 - xq^2)} \rho_0(x; q). \quad (4.74)$$

Note that in calculating $\rho_{1,N}(x; q)$, the normalization of e_0^V is irrelevant, and only the fact that $e_0^V \propto E_{41}$ is needed. The complete R -matrix satisfying (4.25) is therefore (setting $\rho_0(x; q) = 1$)

$$\check{R}(x; q) = \mathcal{P}_0(q) + \left(\frac{x - q^2}{1 - xq^2} \right) \mathcal{P}_1(q) + 2q^2 \frac{q^2 - 1}{\sqrt{1 + q^4}} \frac{1 - x^2}{(x - q^2)(1 - xq^2)} \mathcal{P}_N(q), \quad (4.75)$$

and is unique up to multiplication by a scalar factor. One can easily show that

$$\check{R}(1; q) = 1, \quad \check{R}(x; q)\check{R}(x^{-1}; q) = 1, \quad (4.76)$$

where the latter relation is the unitarity requirement. As stated above, the R -matrix for $U_q[osp(2|2)^{(1)}]$ has been previously computed in [56, 57] and [58]. Our solution agrees with that of [56], where the same $U_q[osp(2|2)]$ simple root system is used. To obtain (4.75) from the more general result presented in [56] (in the limit $b \rightarrow 0$), one needs to take $x \rightarrow x^{-1}$ and $q \rightarrow q^{-1}$ due to the different conventions used.

4d. Crossing symmetry

To go from the R -matrix (4.75) to a physical S -matrix we need to calculate the overall scalar factor $v(x; q)$. This factor is necessary in order to make $S(x; q)$ crossing symmetric.

As it stands, (4.75) satisfies the Yang-Baxter and unitarity requirements. We have seen that these two conditions follow from the intertwining property of $\check{R}(x; q)$ with the comultiplication Δ . In a similar manner, crossing symmetry arises due to an additional “intertwining” property of the universal R -matrix $R(x; q)$ with the antipode operation \mathcal{S} . With this additional property the R -matrix also becomes crossing symmetric, which allows us to identify $\check{R}(x; q)$ with the S -matrix. In terms of the notation (4.18), this means absorbing the scalar factor into $\check{R}(x; q)$, $v(x; q)\check{R}(x; q) \rightarrow \check{R}(x; q)$, setting $S(x; q) = \check{R}(x; q) = PR(x; q)$, and requiring this new R -matrix to satisfy the crossing constraint (4.92) along with (4.25). In the general discussion below, we will make this change taking $S(x; q) = \check{R}(x; q)$. We begin by reviewing the antipode operation. From its relation to the universal R -matrix we will then derive the crossing symmetry constraint. (For further details the reader is referred to [52].)

The antipode is one of the defining structures of a Hopf algebra which acts to connect the multiplication and comultiplication operations. For a non-affine quantum supergroup $U_q(g)$, the antipode \mathcal{S} is defined by its action on the generators as follows [49] ($i = 1, 2, \dots, r$)

$$\begin{aligned} \pi_V(\mathcal{S}(e_i)) & = -q^{-\alpha_i \cdot \alpha_i/2} \pi_V(e_i) \\ \pi_V(\mathcal{S}(f_i)) & = -q^{+\alpha_i \cdot \alpha_i/2} \pi_V(f_i) \\ \pi_V(\mathcal{S}(h_i)) & = -\pi_V(h_i), \end{aligned} \quad (4.77)$$

where the notation $\pi_V(\cdot)$ means all quantities are taken in the representation V . (In the notation of section 3d, $\pi_V(e_i) = e_i^V$, etc..) For $q = 1$ this reduces to the classical antipode \mathcal{S}_{cl}

$$\pi_V(\mathcal{S}_{cl}(a)) = -\pi_V(a), \quad a \in \{e_i, f_i, h_i\}. \quad (4.78)$$

The antipode can be extended to a graded anti-automorphism, so that for homogeneous elements a and b (i.e., elements with definite parity)

$$\mathcal{S}(ab) = (-1)^{d(a)d(b)} \mathcal{S}(b)\mathcal{S}(a). \quad (4.79)$$

To build the crossing relation a concept of charge conjugation is required, whereby a particle is transformed into its antiparticle. At the more formal level, the charge conjugation operation is expressed in terms of a charge conjugation matrix. If there exists a matrix C satisfying

$$\pi_V(\mathcal{S}(a)) = C^{-1} (\pi_V(a))^{st} C, \quad a \in \{e_i, f_i, h_i\}, \quad (4.80)$$

then C is defined to be the charge conjugation matrix. In the classical case this becomes

$$-\pi_V(a) = C^{-1} (\pi_V(a))^{st} C, \quad a \in \{e_i, f_i, h_i\}. \quad (4.81)$$

Here st denotes the supertranspose, which for homogeneous elements is given by

$$(\pi_V(a))_{\alpha\beta}^{st} = (-1)^{d(\alpha)d(\beta)+d(\beta)} (\pi_V(a))_{\beta\alpha}. \quad (4.82)$$

For the $U_q[osp(2|2)]$ simple root system (3.21), we have $\alpha_1 \cdot \alpha_1 = \alpha_2 \cdot \alpha_2 = 0$, and one can show that in the fundamental representation (3.59)

$$C = \begin{pmatrix} & & \pm 1 \\ & 1 & \\ & & 1 \\ \mp 1 & & \end{pmatrix}. \quad (4.83)$$

The signs refer to the two different parity assignments (see section 4c).

For an affine quantum supergroup $U_q(\hat{g})$, the antipode is still given by (4.77). However, in order to correctly define a charge conjugation matrix it is necessary to shift the spectral parameter. Thus it is convenient to explicitly display the θ dependence ($i = 0, 1, \dots, r$)

$$\begin{aligned} \pi_V^\theta(\mathcal{S}(e_i)) &= -q^{-\alpha_i \cdot \alpha_i / 2} e^{+\theta s_i} \pi_V(e_i) \\ \pi_V^\theta(\mathcal{S}(f_i)) &= -q^{+\alpha_i \cdot \alpha_i / 2} e^{-\theta s_i} \pi_V(f_i) \\ \pi_V^\theta(\mathcal{S}(h_i)) &= -\pi_V(h_i). \end{aligned} \quad (4.84)$$

Here the notation $\pi_V^\theta(\cdot)$ indicates the θ dependence and s_i is the Lorentz spin of e_i , i.e.,

$$\pi_V^\theta(e_i) = x_i \pi_V(e_i) = x_i e_i^V, \quad \pi_V^\theta(f_i) = x_i^{-1} \pi_V(f_i) = x_i^{-1} f_i^V, \quad x_i = e^{\theta s_i}, \quad (4.85)$$

gives the affinized representation of $U_q(\hat{g})$ on a loop algebra in some gradation (s_0, s_1, \dots, s_r) . The e_i^V, f_i^V and h_i^V are representation matrices, as in (3.59), (3.61b) and (3.62) for $U_q[osp(2|2)^{(1)}]$. In the homogeneous gradation we are working with $s_0 = -2/\hat{\gamma}, s_1 = s_2 = 0$, giving

$$\pi_V^\theta(e_{1,2}) = e_{1,2}^V, \quad \pi_V^\theta(e_0) = x e_0^V \quad (x = e^{-2\theta/\hat{\gamma}}), \quad (4.86)$$

and so forth.

If (4.84) can now be written as

$$\begin{aligned} \pi_V^\theta(\mathcal{S}(e_i)) &= e^{+(i\pi+\theta)s_i} C^{-1} (\pi_V(e_i))^{st} C = C^{-1} (\pi_V^{\theta+i\pi}(e_i))^{st} C \\ \pi_V^\theta(\mathcal{S}(f_i)) &= e^{-(i\pi+\theta)s_i} C^{-1} (\pi_V(f_i))^{st} C = C^{-1} (\pi_V^{\theta+i\pi}(f_i))^{st} C \\ \pi_V^\theta(\mathcal{S}(h_i)) &= C^{-1} (\pi_V(e_i))^{st} C = C^{-1} (\pi_V^{\theta+i\pi}(h_i))^{st} C, \end{aligned} \quad (4.87)$$

for some matrix C , then this defines C to be the (affine) charge conjugation matrix. Specializing to the algebra $U_q[osp(2|2)^{(1)}]$, we have

$$q = -e^{-i\pi/\hat{\gamma}} \quad (4.88)$$

$$q^{\mp\alpha_1 \cdot \alpha_1 / 2} = q^{\mp\alpha_2 \cdot \alpha_2 / 2} = 1, \quad q^{\mp\alpha_0 \cdot \alpha_0 / 2} = e^{\mp 2i\pi/\hat{\gamma}}, \quad (4.89)$$

giving for (4.84)

$$\pi_V^\theta(\mathcal{S}(e_{1,2})) = -\pi_V(e_{1,2})$$

$$\begin{aligned}
\pi_V^\theta(\mathcal{S}(e_0)) &= -e^{+(\theta+i\pi)(-2/\hat{\gamma})} \pi_V(e_0) \\
\pi_V^\theta(\mathcal{S}(f_{1,2})) &= -\pi_V(f_{1,2}) \\
\pi_V^\theta(\mathcal{S}(f_0)) &= -e^{-(\theta+i\pi)(-2/\hat{\gamma})} \pi_V(f_0) \\
\pi_V^\theta(\mathcal{S}(h_{0,1,2})) &= -\pi_V(h_{0,1,2}).
\end{aligned} \tag{4.90}$$

Comparing (4.90) with (4.87), we see that C has to satisfy (4.81) for all the affine generators. One can check the same $U_q[osp(2|2)]$ conjugation matrix (4.83) also satisfies (4.81) for the fundamental affine representation (3.59), (3.61b) and (3.62). (The fact that the charge conjugation matrix is the same for $U_q[osp(2|2)]$ and $U_q[osp(2|2)^{(1)}]$ is a consequence of the specific representation, gradation and root system we are working with. In general this is not the case.)

The crossing relation can now be derived from the following relation between the universal R -matrix and the antipode

$$(\mathcal{S} \otimes 1)R = R^{-1}. \tag{4.91}$$

Evaluating this in the representation $\pi_V^{\theta_1} \otimes \pi_V^{\theta_2}$, and making use of (4.87) one gets $(\theta = \theta_1 - \theta_2)$

$$R(\theta; \hat{\gamma})(C^{-1} \otimes 1)(R(i\pi + \theta; \hat{\gamma}))^{st_1}(C \otimes 1) = 1, \tag{4.92}$$

where st_1 means taking the supertranspose in only the first space of the tensor product $V \otimes V$. In components we have

$$(R^{st_1})_{a_1 a_2}^{b_1 b_2} = (-1)^{d(a_1)d(b_1)+d(a_1)} R_{b_1 a_2}^{a_1 b_2}. \tag{4.93}$$

We also have written $(\theta; \hat{\gamma})$ instead of $(x; q)$ for the variable dependence of R and will freely use both notations. Equation (4.92) is easily derived using the general expression for R obtained via the quantum double construction [59, 60]

$$R = \sum_i a_i \otimes a^i. \tag{4.94}$$

Substituting $S(\theta; \hat{\gamma}) = PR(\theta; \hat{\gamma})$ into (4.92) gives the crossing relation for the S -matrix

$$PS(\theta; \hat{\gamma})(C^{-1} \otimes 1)(PS(i\pi + \theta; \hat{\gamma}))^{st_1}(C \otimes 1) = 1. \tag{4.95}$$

With the unitarity condition

$$S(\theta; \hat{\gamma})S(-\theta; \hat{\gamma}) = 1, \tag{4.96}$$

equation (4.95) can be rewritten as

$$S(\theta; \hat{\gamma}) = (C^{-1} \otimes 1)(PS(i\pi - \theta; \hat{\gamma}))^{st_1}(C \otimes 1)P. \tag{4.97}$$

We have derived (4.97) rather formally, though its physical interpretation is simple: the amplitude for the direct-channel process

$$|a_1, \theta_1\rangle \otimes |a_2, \theta_2\rangle \longrightarrow |b_2, \theta_2\rangle \otimes |b_1, \theta_1\rangle, \tag{4.98}$$

is the same as the amplitude for the cross-channel process

$$|a_2, \theta_2 + i\pi/2\rangle \otimes |\bar{b}_1, \theta_1 - i\pi/2\rangle \longrightarrow |\bar{a}_1, \theta_1 - i\pi/2\rangle \otimes |b_2, \theta_2 + i\pi/2\rangle, \tag{4.99}$$

where the overbar denotes the conjugated state as determined by the charge conjugation matrix.

4e. The minimal S -matrix

We now want to build a crossing symmetric S -matrix starting from the previously obtained result

$$S(x; q) = v(x; q)\check{R}(x; q) = v(x; q)PR(x; q). \tag{4.100}$$

Here we have returned to the notation of section 4c, with $\check{R}(x; q)$ being the specific solution (4.75) and not the crossing symmetric R -matrix satisfying (4.92). In general, a solution for $v(x; q)$ making $S(x; q)$ crossing symmetric will spoil unitarity. (Recall $\check{R}(x; q)$ is unitary (4.76).) Thus the crossing and unitarity equations need to be considered together when determining $v(x; q)$.

The constraints (4.96) and (4.95) give the following equations for $v(x; q)$

$$v(x; q)v(x^{-1}; q) = 1 \quad (4.101)$$

$$v(x; q)v(xq^2; q) [R(x; q)(C^{-1} \otimes 1)(R(xq^2; q))^{st_1}(C \otimes 1)] = 1, \quad (4.102)$$

where we have used

$$x(\theta) = e^{-2\theta/\hat{\gamma}}, \quad x(i\pi + \theta) = x(\theta)q^2. \quad (4.103)$$

Evaluating the quantity in square brackets, $[\dots]$, we get

$$[\dots] = \frac{(x - q^2)}{(x - 1)} \frac{(1 - xq^2)}{(1 - xq^4)}, \quad (4.104)$$

for both parity assignments. Therefore the functional relations determining the scalar factor are

$$v(x; q)v(x^{-1}; q) = 1 \quad (4.105a)$$

$$v(x; q)v(xq^2; q) = \frac{(x - 1)}{(x - q^2)} \frac{(1 - xq^4)}{(1 - xq^2)}. \quad (4.105b)$$

We choose to express these equations in a slightly different form. Introduce $\hat{v}(x; q)$ via

$$v(x; q) = \frac{1}{4\pi^2} \frac{1}{xq^2} (x - q^2)(1 - xq^2)\hat{v}(x; q). \quad (4.106)$$

Then (4.105a) and (4.105b) are equivalent to

$$\hat{v}(x; q)\hat{v}(x^{-1}; q) = \frac{(4\pi^2 q^2)^2}{(1 - xq^2)^2(1 - x^{-1}q^2)^2} \quad (4.107)$$

$$\hat{v}(x^{-1}q^2; q) = \hat{v}(x; q), \quad (4.107b)$$

or in terms of $(\theta, \hat{\gamma})$

$$\hat{v}(\theta; \hat{\gamma})\hat{v}(-\theta; \hat{\gamma}) = \frac{\pi^4}{\sin^2\left(\frac{1}{\hat{\gamma}}(\pi + i\theta)\right)\sin^2\left(\frac{1}{\hat{\gamma}}(\pi - i\theta)\right)} \quad (4.108a)$$

$$\hat{v}(i\pi - \theta; \hat{\gamma}) = \hat{v}(\theta; \hat{\gamma}). \quad (4.108b)$$

A solution to these equations can be constructed iteratively as discussed in [25, 34]. One begins with a specific solution of (4.108a), call it \hat{v}_0 . This will in general not be a solution of (4.108b). One then looks for a solution of (4.108b), denoted \hat{v}_1 , that is of the form $\hat{v}_1 = \hat{v}_0 f_1$ for some function f_1 . Now \hat{v}_1 will no longer satisfy (4.108a) and one has to re-construct a unitary solution. In order not to end up with the prior solution \hat{v}_0 , a new solution \hat{v}_2 , of the form $\hat{v}_2 = \hat{v}_1 f_2 = \hat{v}_0 f_1 f_2$, is sought. The process is then repeated, eventually giving a solution in the form of an infinite product $\hat{v} = \hat{v}_0 \prod_i f_i$ (assuming everything converges). This recursive method will become clear as we solve (4.108). The final solution for v will depend on the choice of \hat{v}_0 . This reflects the fact that (4.108) does not have a unique solution. However all solutions will differ only by a product (possibly infinite) of CDD factors. For a certain choice of \hat{v}_0 the solution will be minimal, meaning that it will contain a minimum number of poles in the physical strip $0 < \text{Im } \theta < \pi$. We will build this minimal solution.

In constructing a minimal solution it is more convenient, and perhaps even necessary, to work with (4.108) rewritten in terms of gamma functions as

$$\hat{v}(\theta; \hat{\gamma})\hat{v}(-\theta; \hat{\gamma}) = \left(\Gamma\left(1 - \frac{1}{\hat{\gamma}}\left(1 + \frac{i\theta}{\pi}\right)\right) \Gamma\left(\frac{1}{\hat{\gamma}}\left(1 + \frac{i\theta}{\pi}\right)\right) \Gamma\left(1 - \frac{1}{\hat{\gamma}}\left(1 - \frac{i\theta}{\pi}\right)\right) \Gamma\left(\frac{1}{\hat{\gamma}}\left(1 - \frac{i\theta}{\pi}\right)\right) \right)^2 \quad (4.109a)$$

$$\hat{v}(i\pi - \theta; \hat{\gamma}) = \hat{v}(\theta; \hat{\gamma}). \quad (4.109b)$$

For an initial solution of (4.109a) we take

$$\hat{v}_0(\theta; \hat{\gamma}) = - \left(\Gamma \left(1 - \frac{1}{\hat{\gamma}} \left(1 + \frac{i\theta}{\pi} \right) \right) \Gamma \left(\frac{1}{\hat{\gamma}} \left(1 - \frac{i\theta}{\pi} \right) \right) \right)^2. \quad (4.110)$$

However, this does not solve (4.109b). So we adjust (4.110) as follows

$$\hat{v}_1(\theta; \hat{\gamma}) = \hat{v}_0(\theta; \hat{\gamma}) f_1(\theta; \hat{\gamma}), \quad (4.111)$$

where

$$f_1(\theta; \hat{\gamma}) = \hat{v}_0(i\pi - \theta; \hat{\gamma}). \quad (4.112)$$

Now $\hat{v}_1(\theta; \hat{\gamma})$ is crossing symmetric but spoils the unitarity constraint. Unitarity is restored by taking

$$\hat{v}_2(\theta; \hat{\gamma}) = \hat{v}_0(\theta; \hat{\gamma}) f_1(\theta; \hat{\gamma}) f_2(\theta; \hat{\gamma}) \quad (4.113)$$

where

$$f_2(\theta; \hat{\gamma}) = \frac{1}{f_1(-\theta; \hat{\gamma})}. \quad (4.114)$$

At the next step we have, restoring crossing symmetry,

$$\hat{v}_3(\theta; \hat{\gamma}) = \hat{v}_0(\theta; \hat{\gamma}) f_1(\theta; \hat{\gamma}) f_2(\theta; \hat{\gamma}) f_3(\theta; \hat{\gamma}) \quad (4.115)$$

where

$$f_3(\theta; \hat{\gamma}) = f_2(i\pi - \theta; \hat{\gamma}), \quad (4.116)$$

and so forth, with the process never terminating. From the structure of the \hat{v}_i 's, we see that the complete solution to (4.109) takes the form

$$\hat{v}(\theta; \hat{\gamma}) = \hat{v}_0(\theta; \hat{\gamma}) \prod_{n=1}^{\infty} \frac{\hat{v}_0((2n-1)i\pi - \theta; \hat{\gamma}) \hat{v}_0(2ni\pi + \theta; \hat{\gamma})}{\hat{v}_0((2n-1)i\pi + \theta; \hat{\gamma}) \hat{v}_0(2ni\pi - \theta; \hat{\gamma})}. \quad (4.117)$$

Explicitly we have

$$\hat{v}(\theta; \hat{\gamma}) = -\Gamma^2 \left(\frac{1}{\hat{\gamma}} \left(1 + \frac{i\theta}{\pi} \right) \right) \Gamma^2 \left(1 - \frac{1}{\hat{\gamma}} \left(1 + \frac{i\theta}{\pi} \right) \right) \left(\frac{\Gamma \left(1 + \frac{1}{\hat{\gamma}} \frac{i\theta}{\pi} \right)}{\Gamma \left(1 - \frac{1}{\hat{\gamma}} \frac{i\theta}{\pi} \right)} \right)^2 (I(\theta; \hat{\gamma}))^2, \quad (4.118)$$

where

$$I(\theta; \hat{\gamma}) = \prod_{n=1}^{\infty} \frac{\Gamma \left(2n \frac{1}{\hat{\gamma}} + \frac{1}{\hat{\gamma}} \frac{i\theta}{\pi} \right) \Gamma \left(1 + 2n \frac{1}{\hat{\gamma}} + \frac{1}{\hat{\gamma}} \frac{i\theta}{\pi} \right) \Gamma \left((2n-1) \frac{1}{\hat{\gamma}} - \frac{1}{\hat{\gamma}} \frac{i\theta}{\pi} \right) \Gamma \left(1 + (2n-1) \frac{1}{\hat{\gamma}} - \frac{1}{\hat{\gamma}} \frac{i\theta}{\pi} \right)}{\Gamma \left(2n \frac{1}{\hat{\gamma}} - \frac{1}{\hat{\gamma}} \frac{i\theta}{\pi} \right) \Gamma \left(1 + 2n \frac{1}{\hat{\gamma}} - \frac{1}{\hat{\gamma}} \frac{i\theta}{\pi} \right) \Gamma \left((2n-1) \frac{1}{\hat{\gamma}} + \frac{1}{\hat{\gamma}} \frac{i\theta}{\pi} \right) \Gamma \left(1 + (2n-1) \frac{1}{\hat{\gamma}} + \frac{1}{\hat{\gamma}} \frac{i\theta}{\pi} \right)}. \quad (4.119)$$

The factor $I(\theta; \hat{\gamma})$ is identical to the infinite product that appears in the sine-Gordon S -matrix [7]. In the notation of [7] (identifying $1/\hat{\gamma} = 8\pi/\gamma$)

$$I(\theta; \hat{\gamma}) = \prod_{n=1}^{\infty} R_n(\theta) R_n(i\pi - \theta) \quad (4.120)$$

This also shows that (4.119) converges. Taking into account (4.106), the final expression for $v(\theta; \hat{\gamma})$ that solves (4.105) is

$$v(\theta; \hat{\gamma}) = \frac{\Gamma \left(1 - \frac{1}{\hat{\gamma}} - \frac{1}{\hat{\gamma}} \frac{i\theta}{\pi} \right) \Gamma \left(\frac{1}{\hat{\gamma}} + \frac{1}{\hat{\gamma}} \frac{i\theta}{\pi} \right)}{\Gamma \left(1 - \frac{1}{\hat{\gamma}} + \frac{1}{\hat{\gamma}} \frac{i\theta}{\pi} \right) \Gamma \left(\frac{1}{\hat{\gamma}} - \frac{1}{\hat{\gamma}} \frac{i\theta}{\pi} \right)} \left(\frac{\Gamma \left(1 + \frac{1}{\hat{\gamma}} \frac{i\theta}{\pi} \right)}{\Gamma \left(1 - \frac{1}{\hat{\gamma}} \frac{i\theta}{\pi} \right)} \right)^2 (I(\theta; \hat{\gamma}))^2. \quad (4.121)$$

Therefore, the minimal S -matrix for the Toda model (3.27) is

$$S(\theta; \hat{\gamma}) = v(\theta; \hat{\gamma}) \check{R}(\theta; \hat{\gamma}), \quad (4.122)$$

with $\check{R}(\theta; \hat{\gamma})$ given by (4.75) and $v(\theta; \hat{\gamma})$ as above. The S -matrix is unique up to an arbitrariness only of the CDD type. As previously stated, the CDD factors are determined by applying the bootstrap procedure. In general this is a complicated task and requires knowing all the particle multiplets of the theory. The bootstrap equation constrains the poles and zeros of the various S -matrices associated with the different multiplets. The final result is that the complete S -matrix (for the vector representation) takes the form

$$S_{\text{complete}}(\theta; \hat{\gamma}) = X(\theta; \hat{\gamma}) v(\theta; \hat{\gamma}) \check{R}(\theta; \hat{\gamma}), \quad (4.123)$$

where $X(\theta; \hat{\gamma})$ is a product of CDD type factors.

We give alternative expressions for the $\rho_i(\theta; \hat{\gamma})$ factors (4.72) and (4.74) in terms of gamma functions

$$\rho_1(\theta; \hat{\gamma}) = \frac{\Gamma\left(1 - \frac{1}{\hat{\gamma}} + \frac{1}{\hat{\gamma}} \frac{i\theta}{\pi}\right) \Gamma\left(\frac{1}{\hat{\gamma}} - \frac{1}{\hat{\gamma}} \frac{i\theta}{\pi}\right)}{\Gamma\left(1 - \frac{1}{\hat{\gamma}} - \frac{1}{\hat{\gamma}} \frac{i\theta}{\pi}\right) \Gamma\left(\frac{1}{\hat{\gamma}} + \frac{1}{\hat{\gamma}} \frac{i\theta}{\pi}\right)} \quad (4.124)$$

$$\rho_N(\theta; \hat{\gamma}) = -\frac{2q}{\sqrt{1+q^4}} \frac{\Gamma\left(1 - \frac{1}{\hat{\gamma}} - \frac{1}{\hat{\gamma}} \frac{i\theta}{\pi}\right) \Gamma\left(\frac{1}{\hat{\gamma}} + \frac{1}{\hat{\gamma}} \frac{i\theta}{\pi}\right) \Gamma\left(1 - \frac{1}{\hat{\gamma}} + \frac{1}{\hat{\gamma}} \frac{i\theta}{\pi}\right) \Gamma\left(\frac{1}{\hat{\gamma}} - \frac{1}{\hat{\gamma}} \frac{i\theta}{\pi}\right)}{\Gamma\left(1 - \frac{1}{\hat{\gamma}}\right) \Gamma\left(\frac{1}{\hat{\gamma}}\right) \Gamma\left(1 - \frac{2}{\hat{\gamma}} \frac{i\theta}{\pi}\right) \Gamma\left(\frac{2}{\hat{\gamma}} \frac{i\theta}{\pi}\right)}. \quad (4.125)$$

These expressions are useful for analyzing the pole structure of $S(\theta; \hat{\gamma})$ (see section 5).

4f. The Yangian limit and the $osp(2|2)$ current-current S -matrix

In this section we compute the Yangian symmetric S -matrix, $S_Y(\theta)$, for the original $osp(2|2)$ current-current model (2.3). As explained in section 2, $S_Y(\theta)$ can be obtained from (4.122) by taking the marginal limit $\hat{\beta} \rightarrow 1^+$, or $\epsilon \rightarrow 0^+$ where $\epsilon \equiv 1/\hat{\gamma}$. (This limit is equivalent to the rational limit usually considered for R -matrices, and the resulting R -matrix being referred to as the rational R -matrix.)

The marginal limit is easily evaluated for the various components of $S(\theta; \hat{\gamma})$. For the projectors, which are well defined as $\epsilon \rightarrow 0^+$, we simply set $q = -1$. For the remaining factors, $v(\theta; \hat{\gamma})$ and $\rho_{1,N}(\theta; \hat{\gamma})$, we make use of

$$\lim_{\epsilon \rightarrow 0^+} \left(\frac{\Gamma(\epsilon A)}{\Gamma(\epsilon B)} \right) = \frac{B}{A}. \quad (4.126)$$

One finds

$$\begin{aligned} \lim_{\epsilon \rightarrow 0^+} v(\theta; \hat{\gamma}) &= \frac{1 - \frac{i\theta}{\pi}}{1 + \frac{i\theta}{\pi}} \left(\prod_{n=1}^{\infty} \frac{(2n-1 + \frac{i\theta}{\pi})(2n - \frac{i\theta}{\pi})}{(2n-1 - \frac{i\theta}{\pi})(2n + \frac{i\theta}{\pi})} \right)^2 \\ &= \frac{1 - \frac{i\theta}{\pi}}{1 + \frac{i\theta}{\pi}} \left(\frac{\Gamma(1 + \frac{1}{2} \frac{i\theta}{\pi}) \Gamma(\frac{1}{2} - \frac{1}{2} \frac{i\theta}{\pi})}{\Gamma(1 - \frac{1}{2} \frac{i\theta}{\pi}) \Gamma(\frac{1}{2} + \frac{1}{2} \frac{i\theta}{\pi})} \right)^2 \end{aligned} \quad (4.127)$$

$$\lim_{\epsilon \rightarrow 0^+} \rho_1(\theta; \hat{\gamma}) = \frac{1 + \frac{i\theta}{\pi}}{1 - \frac{i\theta}{\pi}} \quad (4.128)$$

$$\lim_{\epsilon \rightarrow 0^+} \rho_N(\theta; \hat{\gamma}) = 2\sqrt{2} \frac{\frac{i\theta}{\pi}}{(1 + \frac{i\theta}{\pi})(1 - \frac{i\theta}{\pi})}. \quad (4.129)$$

These results could have been obtained directly from (4.72) and (4.74), and by taking the marginal limit before calculating $v(\theta)$. Combining everything, $S_Y(\theta)$ takes the block diagonal form as in (4.61) with the individual blocks being:

(i) One-dimensional blocks:

$$S_Y(\theta) = I_Y^2(\theta), \quad (4.130a)$$

for $|1\rangle \otimes |1\rangle$ and $|4\rangle \otimes |4\rangle$; and

$$S_Y(\theta) = \frac{i\pi + \theta}{i\pi - \theta} I_Y^2(\theta), \quad (4.130b)$$

for $|2\rangle \otimes |2\rangle$ and $|3\rangle \otimes |3\rangle$.

(ii) Two-dimensional blocks:

$$S_Y(\theta) = \frac{1}{i\pi - \theta} \begin{pmatrix} i\pi & \pm\theta \\ \pm\theta & i\pi \end{pmatrix} I_Y^2(\theta), \quad (4.130c)$$

for the four pairs of states

$$(|1\rangle \otimes |2\rangle, |2\rangle \otimes |1\rangle), (|1\rangle \otimes |3\rangle, |3\rangle \otimes |1\rangle), (|2\rangle \otimes |4\rangle, |4\rangle \otimes |2\rangle), (|3\rangle \otimes |4\rangle, |4\rangle \otimes |3\rangle).$$

(ii) Four-dimensional block:

$$S_Y(\theta) = -\frac{1}{(i\pi - \theta)^2} \begin{pmatrix} \pi(\pi + 2i\theta) & \pm i\pi\theta & \pm i\pi\theta & -\theta^2 \\ \mp i\pi\theta & \pi^2 & \theta(\theta - 2i\pi) & \pm i\pi\theta \\ \mp i\pi\theta & \theta(\theta - 2i\pi) & \pi^2 & \pm i\pi\theta \\ -\theta^2 & \mp i\pi\theta & \mp i\pi\theta & \pi(\pi + 2i\theta) \end{pmatrix} I_Y^2(\theta), \quad (4.130d)$$

for the basis states

$$(|1\rangle \otimes |4\rangle, |2\rangle \otimes |3\rangle, |3\rangle \otimes |2\rangle, |4\rangle \otimes |1\rangle).$$

The factor $I_Y(\theta)$ is the $\epsilon \rightarrow 0^+$ limit of $I(\theta; \hat{\gamma})$:

$$I_Y(\theta) = \frac{\Gamma(1 + \frac{i\theta}{2\pi}) \Gamma(\frac{1}{2} - \frac{i\theta}{2\pi})}{\Gamma(1 - \frac{i\theta}{2\pi}) \Gamma(\frac{1}{2} + \frac{i\theta}{2\pi})}. \quad (4.131)$$

This completes our calculation of the S -matrix for the $osp(2|2)$ current-current model (2.3).

We have not yet proven that (4.130) is Yangian symmetric. This means showing $S_Y(\theta)$ commutes with the comultiplication of the Yangian generators. Unfortunately the general theory of super Yangians associated with Lie superalgebras is not as fully developed as that of quantum supergroups. (See [61, 62] and references therein for a discussion of super Yangians.) For ordinary Lie algebras, one can realize a Yangian structure in terms of non-local charges arising from curvature-free currents [36]. This construction is particularly useful in studying Yangian symmetric field theories and S -matrices. However we are unaware of a similar characterization of the Yangians for Lie superalgebras. Thus we do not strictly show (4.130) to be Yangian symmetric but give some supporting evidence. Our discussion is based on a comparison with bosonic Yangian symmetric systems. Examples of such systems include Gross-Neveu type models and the sine-Gordon model. At the marginal point, these models consist of current-current type perturbations of a free field theory. Furthermore, their Yangian symmetric S -matrices can be obtained by taking the marginal limit of a S -matrix with affine quantum group symmetry [34]. (For the Gross-Neveu model this quantum group symmetric S -matrix corresponds to an affine Toda theory.) The Yangian symmetry of the S -matrix, i.e., the vanishing commutator of S with the Yangian comultiplication, follows from the marginal limit of the quantum affine symmetry relations. Here we are dealing with a supersymmetric analog of the bosonic Gross-Neveu model. The current-current model (2.3) is a Gross-Neveu type model based on $osp(2|2)$, and (2.18) is an $osp(2|2)^{(1)}$ Toda-type system. As for the bosonic case, we expect that the marginal S -matrix is Yangian symmetric and the corresponding symmetry relations can be extracted from (4.11).

Let us first recall the bosonic situation. The Yangian $Y(g)$ based on a semi-simple Lie algebra g of rank r is generated by the set of charges $\{Q_a^{(0)}, Q_a^{(1)}\}_{a=1, \dots, r}$ satisfying, among others, the relations [36, 60, 63, 64]

$$\begin{aligned} [Q_a^{(0)}, Q_b^{(0)}] &= f_{abc} Q_c^{(0)} \\ [Q_a^{(0)}, Q_b^{(1)}] &= f_{abc} Q_c^{(0)}, \end{aligned} \quad (4.132)$$

where the f_{abc} 's are structure constants of g . $Y(g)$ can be made into a Hopf algebra with the comultiplication

$$\Delta(Q_a^{(0)}) = Q_a^{(0)} \otimes 1 + 1 \otimes Q_a^{(0)}$$

$$\Delta(Q_a^{(1)}) = Q_a^{(1)} \otimes 1 + 1 \otimes Q_a^{(1)} - \frac{1}{2} f_{abc} Q_b^{(0)} \otimes Q_c^{(0)}. \quad (4.133)$$

The charges have non-trivial Lorentz spin and under a Lorentz boost by θ , denoted T_θ , behave as

$$\begin{aligned} T_\theta(Q_a^{(0)}) &= Q_a^{(0)} \\ T_\theta(Q_a^{(1)}) &= Q_a^{(1)} + c\theta Q_a^{(0)}, \end{aligned} \quad (4.134)$$

where c is a normalization constant independent of the index a . The Yangian symmetric S -matrix then commutes with (4.133) evaluated in the “gradation” given by T_θ

$$S(\theta_1 - \theta_2)(T_{\theta_1} \otimes T_{\theta_2})\Delta(Q_a^{(0,1)}) = (T_{\theta_2} \otimes T_{\theta_1})\Delta(Q_a^{(0,1)})S(\theta_1 - \theta_2), \quad (4.135)$$

or explicitly

$$S(\theta_1 - \theta_2) \left(Q_a^{(0)} \otimes 1 + 1 \otimes Q_a^{(0)} \right) = \left(Q_a^{(0)} \otimes 1 + 1 \otimes Q_a^{(0)} \right) S(\theta_1 - \theta_2), \quad (4.136a)$$

$$\begin{aligned} S(\theta_1 - \theta_2) &\left((Q_a^{(1)} + c\theta_1 Q_a^{(0)}) \otimes 1 + 1 \otimes (Q_a^{(1)} + c\theta_2 Q_a^{(0)}) - \frac{1}{2} f_{abc} Q_b^{(0)} \otimes Q_c^{(0)} \right) \\ &= \left((Q_a^{(1)} + c\theta_2 Q_a^{(0)}) \otimes 1 + 1 \otimes (Q_a^{(1)} + c\theta_1 Q_a^{(0)}) - \frac{1}{2} f_{abc} Q_b^{(0)} \otimes Q_c^{(0)} \right) S(\theta_1 - \theta_2). \end{aligned} \quad (4.136)$$

For the bosonic $su(N)$ Gross-Neveu model, it was shown in [34] that (4.136) can be recovered by expanding the affine quantum group symmetry relations in ϵ , with $\epsilon \rightarrow 0^+$ being the marginal limit. The zeroth order term gives the constraint (4.136a) and the first order term leads to (4.136b).

Now we show that relations similar to (4.136) can also be obtained from (4.11) as $\epsilon \rightarrow 0^+$ ($\epsilon = 1/\hat{\gamma}$). (Note that in writing (4.11) we had canceled the non-zero factors c_i (3.57) from both sides. However as $\epsilon \rightarrow 0^+$, c_i seems to blow up. As explained earlier to make sense of (4.11) we need to regularize the charges by taking $g \rightarrow 0$ such that $g\hat{\gamma}$ is finite.) We will only display the relations for the e_i 's, with those for the other generators treated similarly. The $U_q[osp(2|2)^{(1)}]$ generators will be denoted as e_{iq}, h_{iq} and the ϵ -independent $osp(2|2)^{(1)}$ generators as e_i, h_i . (We also drop the V superscript.) From (3.59), (3.61b) and (3.62) we have to lowest order in ϵ

$$e_{1,2q} = e_{1,2}, \quad h_{iq} = h_i \quad (4.137)$$

$$e_{0q} = ie_0 - \epsilon^2 \frac{\pi^2}{4} ie_0. \quad (4.138)$$

Of course (4.137) is exact since these generators are not deformed. The i factor in (4.138) is a result of the Yangian point being $q = -1$ instead of $q = 1$. Also the spectral parameters are

$$x_{i0} = e^{-2\theta_i \epsilon} \approx 1 - 2\theta_i \epsilon. \quad (4.139)$$

Substituting these expressions into (4.11a) and (4.11b) we find to zeroth order in ϵ

$$S_Y(\theta) \left(e_i \otimes (-1)^{-h_i/2} + (-1)^{-h_i/2} \otimes e_i \right) = \left(e_i \otimes (-1)^{-h_i/2} + (-1)^{-h_i/2} \otimes e_i \right) S_Y(\theta), \quad (i = 0, 1, 2). \quad (4.140)$$

To first order in ϵ we have

$$S_Y(\theta) \Delta_{\theta_1 \theta_2}(e_i) = \Delta_{\theta_2 \theta_1}(e_i) S_Y(\theta), \quad (4.141)$$

where

$$\Delta_{\theta_1 \theta_2}(e_{1,2}) = e_{1,2} \otimes (-1)^{-h_{1,2}/2} h_{1,2} - (-1)^{-h_{1,2}/2} h_{1,2} \otimes e_{1,2} \quad (4.142)$$

$$\Delta_{\theta_1 \theta_2}(e_0) = \frac{4\theta_1}{i\pi} e_0 \otimes (-1)^{-h_0/2} + (-1)^{-h_0/2} \otimes \frac{4\theta_2}{i\pi} e_0 - \left(e_0 \otimes (-1)^{-h_0/2} h_0 - (-1)^{-h_0/2} h_0 \otimes e_0 \right). \quad (4.143)$$

We see that these equations agree with (4.136) if we identify

$$e_i^{(0)} = e_i, \quad h_i^{(0)} = h_i \quad (4.144)$$

$$e_i^{(1)} = 0, \quad (4.145)$$

and define the automorphism T_θ as

$$T_\theta(e_i^{(1)}) = e_i^{(1)} - \frac{2\theta}{i\pi} s_i e_i^{(0)}, \quad (4.146)$$

where $s_i/\hat{\gamma}$ is the Lorentz spin of the charge \overline{Q}^i . Even though the charges $e_i^{(1)}$ are identically zero in this representation, there is non-trivial structure due to the comultiplication (4.142) and (4.143). That the $e_i^{(1)}$'s vanish is not surprising since $U_q[osp(2|2)^{(1)}]$ is not deformed to order ϵ . Unlike (4.134), the θ dependence of $T_\theta(e_i^{(1)})$ is not the same for all generators, which reflects the choice of gradation (homogeneous) and the root structure of $osp(2|2)^{(1)}$. The twisting factors $(-1)^{\pm h_i/2}$ are again a consequence of the $q \rightarrow -1$ limit rather than $q \rightarrow 1$. Off shell, these factors correspond to a choice of statistical Klein factors and are not expected to be dynamical [60]. (Taking $q \rightarrow 1$ gives an equivalent crossing symmetric S -matrix, with the same charge conjugation properties and differing from (4.130) only by a similarity transformation.) The S -matrix (4.130) therefore satisfies symmetry relations analogous to those of (bosonic) Yangian symmetric S -matrices, which supports the claim that $S_Y(\theta)$ has super Yangian symmetry.

5. The analytic structure of $S(\theta; \hat{\gamma})$

We have argued that in the marginal limit the Toda S -matrix $S(\theta; \hat{\gamma})$ gives the minimal S -matrix for the $osp(2|2)$ current-current model (2.3) (in the fundamental vector representation). As $\hat{\beta} \rightarrow 1$, the Toda theory renormalizes to the current-current model and $S(\theta; \hat{\gamma})$ reduces to $S_Y(\theta)$. We now study the pole structure of $S(\theta; \hat{\gamma})$.

In the above calculation of $S(\theta; \hat{\gamma})$ we have taken $\hat{\beta}$ to lie in the range $1 \leq \hat{\beta} < \infty$, which restricts $1/\hat{\gamma}$ to

$$0 \leq \frac{1}{\hat{\gamma}} < 1. \quad (5.1)$$

For these values the S -matrix does not have any poles in the physical strip $0 < \text{Im } \theta < \pi$. In particular, the physical S -matrix $S_Y(\theta)$ does not contain any bound states. This structure is identical to that of the sine-Gordon S -matrix for the range $\sqrt{8\pi} \leq \beta < \infty$ or $-1 < 8\pi/\gamma \leq 0$, with $8\pi/\gamma = 0$ being the Yangian point (though in this range the sine-Gordon model is not well-defined [7]). Of course $S(\theta; \hat{\gamma})$ satisfies the scattering constraints for all values of $1/\hat{\gamma}$ and we made no direct use of (5.1) in deriving $S(\theta; \hat{\gamma})$. For the sine-Gordon model the physically relevant parameter range is $8\pi/\gamma > 0$. Analogously, we will consider the range $1/\hat{\gamma} > 1$. This necessarily means ignoring the relation (3.36) and treating $\hat{\gamma}$ as an independent free parameter. Thus we are now viewing $S(\theta; \hat{\gamma})$ as the fundamental S -matrix for some theory. (Since the quantum group symmetry used to derive $S(\theta; \hat{\gamma})$ is valid only for the range (5.1), we cannot consider $S(\theta; \hat{\gamma})$ to be the S -matrix for the Toda theory (3.27) for $1/\hat{\gamma} > 1$. Also if $1/\hat{\gamma} > 1$, then (3.36) implies $\hat{\beta}$ is purely imaginary, which leads to $\Delta(g) = \overline{\Delta}(g) < 0$, i.e., the perturbation becomes irrelevant. Even so, it is interesting to study the pole structure of $S(\theta; \hat{\gamma})$ independent of any Lagrangian formulation.) For simplicity we will restrict our analysis to generic values of $\hat{\gamma}$ such that $q \neq \pm 1, \pm i$, i.e., we take $1/\hat{\gamma} > 1$ with $1/\hat{\gamma}$ not an integer or half-integer.

The θ -dependence is contained in the factors $v(\theta; \hat{\gamma})$, $\rho_1(\theta; \hat{\gamma})$ and $\rho_N(\theta; \hat{\gamma})$. From (4.119) one can check that the infinite product $I(\theta; \hat{\gamma})$ contains no zeros or poles in the physical strip. In the remaining finite number of gamma functions, there are only two with zeros or poles in the physical strip

$$\Gamma\left(1 + \frac{1}{\hat{\gamma}} \frac{i\theta}{\pi}\right) : \quad \text{has simple poles at } \theta = i\pi\hat{\gamma}(m+1); m = 0, 1, \dots, < \frac{1}{\hat{\gamma}} - 1 \quad (5.2)$$

and

$$\Gamma\left(1 - \frac{1}{\hat{\gamma}} - \frac{1}{\hat{\gamma}} \frac{i\theta}{\pi}\right) : \quad \text{has simple poles at } \theta = i\pi - i\pi\hat{\gamma}(m+1); m = 0, 1, \dots, < \frac{1}{\hat{\gamma}} - 1. \quad (5.3)$$

Combining this with (4.121), (4.124) and (4.125), we find the following pole structure for the non-zero amplitudes (in all cases $m = 0, 1, \dots, < 1/\hat{\gamma} - 1$):

(i) amplitudes $S_{11}^{11} = S_{44}^{44}$:

(a) double poles at $\theta = i\pi\hat{\gamma}(m+1)$ with (omitting the indices)

$$S(\theta; \hat{\gamma}) \sim -\frac{(\pi\hat{\gamma})^2 I_0^m(\hat{\gamma})}{(m!(m+1)!)^2} \frac{1}{(\theta - \theta_0^m)^2}, \quad (5.4)$$

where

$$\theta_0^m = i\pi\hat{\gamma}(m+1), \quad I_0^m(\hat{\gamma}) = I(\theta = \theta_0^m; \hat{\gamma}), \quad (5.5)$$

(b) no poles at $\theta = i\pi - i\pi\hat{\gamma}(m+1)$.

(ii) amplitudes $S_{22}^{22} = S_{33}^{33}$:

(a) double poles at $\theta = i\pi\hat{\gamma}(m+1)$ with

$$S(\theta; \hat{\gamma}) \sim -\frac{(\pi\hat{\gamma})^2 I_0^m(\hat{\gamma})}{(m!(m+1)!)^2} \frac{1}{(\theta - \theta_0^m)^2}, \quad (5.6a)$$

(b) single poles at $\theta = i\pi - i\pi\hat{\gamma}(m+1)$ with

$$S(\theta; \hat{\gamma}) \sim -i\hat{\gamma} \sin(2\pi/\hat{\gamma}) \left(\frac{\Gamma\left(2 - \frac{1}{\hat{\gamma}} + m\right)}{\Gamma\left(\frac{1}{\hat{\gamma}} - m\right)} \right)^2 I_1^m(\hat{\gamma}) \frac{1}{(\theta - \theta_1^m)}, \quad (5.6b)$$

where

$$\theta_1^m = i\pi - i\pi\hat{\gamma}(m+1), \quad I_1^m(\hat{\gamma}) = I(\theta = \theta_1^m; \hat{\gamma}). \quad (5.7)$$

Using

$$I(i\pi - \theta; \hat{\gamma}) = I(\theta; \hat{\gamma}) \left(\frac{\Gamma\left(1 + \frac{1}{\hat{\gamma}} + \frac{1}{\hat{\gamma}} \frac{i\theta}{\pi}\right) \Gamma\left(\frac{1}{\hat{\gamma}} + \frac{1}{\hat{\gamma}} \frac{i\theta}{\pi}\right)}{\Gamma\left(1 - \frac{1}{\hat{\gamma}} \frac{i\theta}{\pi}\right) \Gamma\left(-\frac{1}{\hat{\gamma}} \frac{i\theta}{\pi}\right)} \right)^2, \quad (5.8)$$

expression (5.6b) can be rewritten as

$$S(\theta; \hat{\gamma}) \sim -\frac{2i\pi^2 \hat{\gamma} I_0^m(\hat{\gamma})}{(m!(m+1)!)^2} \cot(\pi/\hat{\gamma}) \frac{1}{(\theta - \theta_1^m)}. \quad (5.6c)$$

(iii) amplitudes $S_{12}^{12} = S_{13}^{13} = S_{24}^{24} = S_{34}^{34}$:

(a) double poles at $\theta = i\pi\hat{\gamma}(m+1)$ with

$$S(\theta; \hat{\gamma}) \sim -\frac{(\pi\hat{\gamma})^2 I_0^m(\hat{\gamma})}{(m!(m+1)!)^2} \frac{1}{(\theta - \theta_0^m)^2}, \quad (5.9a)$$

(b) single poles at $\theta = i\pi - i\pi\hat{\gamma}(m+1)$ with

$$S(\theta; \hat{\gamma}) \sim -\frac{i\pi^2 \hat{\gamma} I_0^m(\hat{\gamma})}{(m!(m+1)!)^2} (\cot(\pi/\hat{\gamma}) - i) \frac{1}{(\theta - \theta_1^m)}. \quad (5.9b)$$

(iv) amplitudes $S_{21}^{21} = S_{31}^{31} = S_{42}^{42} = S_{43}^{43}$:

(a) double poles at $\theta = i\pi\hat{\gamma}(m+1)$ with

$$S(\theta; \hat{\gamma}) \sim -\frac{(\pi\hat{\gamma})^2 I_0^m(\hat{\gamma})}{(m!(m+1)!)^2} \frac{1}{(\theta - \theta_0^m)^2}, \quad (5.10a)$$

(b) single poles at $\theta = i\pi - i\pi\hat{\gamma}(m+1)$ with

$$S(\theta; \hat{\gamma}) \sim -\frac{i\pi^2 \hat{\gamma} I_0^m(\hat{\gamma})}{(m!(m+1)!)^2} (\cot(\pi/\hat{\gamma}) + i) \frac{1}{(\theta - \theta_1^m)}. \quad (5.10b)$$

(v) amplitudes $S_{21}^{12} = S_{12}^{21} = S_{31}^{13} = S_{13}^{31} = S_{42}^{24} = S_{24}^{42} = S_{43}^{34} = S_{34}^{43}$.

(a) single poles at $\theta = i\pi\hat{\gamma}(m+1)$ with

$$S(\theta; \hat{\gamma}) \sim \pm \frac{i\pi^2 \hat{\gamma} I_0^m(\hat{\gamma})}{(m!(m+1)!)^2} \csc(\pi/\hat{\gamma}) \frac{1}{(\theta - \theta_0^m)}, \quad (5.11a)$$

(b) single poles at $\theta = i\pi - i\pi\hat{\gamma}(m+1)$ with

$$S(\theta; \hat{\gamma}) \sim \mp \frac{i\pi^2 \hat{\gamma} I_0^m(\hat{\gamma})}{(m!(m+1)!)^2} \csc(\pi/\hat{\gamma}) \frac{1}{(\theta - \theta_1^m)}. \quad (5.11b)$$

(vi) amplitude S_{14}^{14} :

(a) double poles at $\theta = i\pi\hat{\gamma}(m+1)$ with

$$S(\theta; \hat{\gamma}) \sim - \frac{(\pi\hat{\gamma})^2 I_0^m(\hat{\gamma})}{(m!(m+1)!)^2} \frac{1}{(\theta - \theta_0^m)^2}, \quad (5.12a)$$

(b) double poles at $\theta = i\pi - i\pi\hat{\gamma}(m+1)$ with

$$S(\theta; \hat{\gamma}) \sim \frac{(\pi\hat{\gamma})^2 I_0^m(\hat{\gamma})}{(m!(m+1)!)^2} \frac{1}{(\theta - \theta_1^m)^2}. \quad (5.12b)$$

(vii) amplitude S_{41}^{41} :

(a) double poles at $\theta = i\pi\hat{\gamma}(m+1)$ with

$$S(\theta; \hat{\gamma}) \sim - \frac{(\pi\hat{\gamma})^2 I_0^m(\hat{\gamma})}{(m!(m+1)!)^2} \frac{1}{(\theta - \theta_0^m)^2}, \quad (5.13a)$$

(b) double poles at $\theta = i\pi - i\pi\hat{\gamma}(m+1)$ with

$$S(\theta; \hat{\gamma}) \sim \frac{(\pi\hat{\gamma})^2 I_0^m(\hat{\gamma})}{(m!(m+1)!)^2} \frac{1}{(\theta - \theta_1^m)^2}. \quad (5.13b)$$

(viii) amplitudes $S_{23}^{23} = S_{32}^{32}$:

(a) double poles at $\theta = i\pi\hat{\gamma}(m+1)$ with

$$S(\theta; \hat{\gamma}) \sim - \frac{(\pi\hat{\gamma})^2 I_0^m(\hat{\gamma})}{(m!(m+1)!)^2} \frac{1}{(\theta - \theta_0^m)^2}, \quad (5.14a)$$

(b) double poles at $\theta = i\pi - i\pi\hat{\gamma}(m+1)$ with

$$S(\theta; \hat{\gamma}) \sim - \frac{(\pi\hat{\gamma})^2 I_0^m(\hat{\gamma})}{(m!(m+1)!)^2} \frac{1}{(\theta - \theta_1^m)^2}. \quad (5.14b)$$

(ix) amplitudes $S_{23}^{14} = S_{32}^{14} = -S_{14}^{23} = -S_{41}^{32}$:

(a) single poles at $\theta = i\pi\hat{\gamma}(m+1)$ with (here $S = S_{23}^{14}$)

$$S(\theta; \hat{\gamma}) \sim \mp \frac{i\pi^2 \hat{\gamma} I_0^m(\hat{\gamma})}{(m!(m+1)!)^2} (\cot(\pi/\hat{\gamma}) + i) \frac{1}{(\theta - \theta_0^m)}, \quad (5.15a)$$

(b) double poles at $\theta = i\pi - i\pi\hat{\gamma}(m+1)$ with

$$S(\theta; \hat{\gamma}) \sim \pm \frac{(\pi\hat{\gamma})^2 I_0^m(\hat{\gamma})}{(m!(m+1)!)^2} \frac{1}{(\theta - \theta_1^m)^2}. \quad (5.15b)$$

(x) amplitudes $S_{23}^{41} = S_{32}^{41} = -S_{41}^{23} = -S_{14}^{32}$:

(a) single poles at $\theta = i\pi\hat{\gamma}(m+1)$ with (here $S = S_{23}^{41}$)

$$S(\theta; \hat{\gamma}) \sim \pm \frac{i\pi^2 \hat{\gamma} I_0^m(\hat{\gamma})}{(m!(m+1)!)^2} (\cot(\pi/\hat{\gamma}) - i) \frac{1}{(\theta - \theta_0^m)}, \quad (5.16a)$$

(b) double poles at $\theta = i\pi - i\pi\hat{\gamma}(m+1)$ with

$$S(\theta; \hat{\gamma}) \sim \mp \frac{(\pi\hat{\gamma})^2 I_0^m(\hat{\gamma})}{(m!(m+1)!)^2} \frac{1}{(\theta - \theta_1^m)^2}. \quad (5.16b)$$

(xi) amplitudes $S_{32}^{23} = S_{23}^{32}$:

(a) single poles at $\theta = i\pi\hat{\gamma}(m+1)$:

$$S(\theta; \hat{\gamma}) \sim \frac{2i\pi^2 \hat{\gamma} I_0^m(\hat{\gamma})}{(m!(m+1)!)^2} \cot(\pi/\hat{\gamma}) \frac{1}{(\theta - \theta_0^m)}, \quad (5.17a)$$

(b) double poles at $\theta = i\pi - i\pi\hat{\gamma}(m+1)$ with

$$S(\theta; \hat{\gamma}) \sim -\frac{(\pi\hat{\gamma})^2 I_0^m(\hat{\gamma})}{(m!(m+1)!)^2} \frac{1}{(\theta - \theta_1^m)^2}. \quad (5.17b)$$

(xii) amplitudes $S_{41}^{14} = S_{14}^{41}$:

(a) no poles at $\theta = i\pi\hat{\gamma}(m+1)$,

(b) double poles at $\theta = i\pi - i\pi\hat{\gamma}(m+1)$ with

$$S(\theta; \hat{\gamma}) \sim -\frac{(\pi\hat{\gamma})^2 I_0^m(\hat{\gamma})}{(m!(m+1)!)^2} \frac{1}{(\theta - \theta_1^m)^2}. \quad (5.18)$$

These poles and expansions for $S(\theta; \hat{\gamma})$ agree with the crossing constraint, which in components takes the form

$$S_{a_1 a_2}^{b_2 b_1}(\theta; \hat{\gamma}) = (-1)^{\delta_{a_1 4} + \delta_{b_1 4}} S_{a_2 \bar{b}_1}^{\bar{a}_1 b_2}(i\pi - \theta; \hat{\gamma}), \quad \text{if } 1, 4 \text{ are even and } 2, 3 \text{ are odd} \quad (5.19a)$$

$$S_{a_1 a_2}^{b_2 b_1}(\theta; \hat{\gamma}) = (-1)^{\delta_{a_1 1} + \delta_{b_1 1}} S_{a_2 \bar{b}_1}^{\bar{a}_1 b_2}(i\pi - \theta; \hat{\gamma}), \quad \text{if } 1, 4 \text{ are odd and } 2, 3 \text{ are even.} \quad (5.19b)$$

The $(-1)^\delta$ factor is due to the negative sign in the charge conjugation matrix and the bar denotes the conjugated state ($\bar{1} = 4$, etc.). Equation (5.19) implies that for every pole at θ in the direct-channel there is a corresponding pole in the cross-channel.

The amplitudes (vi)-(xii) correspond to transitions between states with zero topological charge, $(T^1, T^2) = (0, 0)$. Simple poles in these amplitudes at $\theta = i\pi\hat{\gamma}(m+1)$ can be interpreted as charge neutral bound states in the direct-channel. These are the “breathers” of the theory. The associated cross-channel poles occur at $\theta = i\pi - i\pi\hat{\gamma}(m+1)$ in (ii) - (iv). There are also bound states of charge $(+2, 0), (-2, 0), (0, +2)$ and $(0, -2)$. These “breathing solitons” appear as simple poles in both the direct- and cross-channels in (v). Lastly, there are various double poles which probably have an explanation in terms of a Coleman-Thun type mechanism [65].

6. Conclusions

We have computed the S -matrix for a certain disordered system. After disorder averaging, the theory can be written as a current-current perturbation of an $osp(2|2)$ supersymmetric CFT. This current-current model is known to be Yangian symmetric. Instead of directly constructing the Yangian symmetric S -matrix, we followed the approach of [34]. This approach consisted of working with a Toda-type theory which renormalizes to the $osp(2|2)$ current-current model at the marginal point. For the Toda theory we built quantum group charges satisfying the $U_q[osp(2|2)^{(1)}]$ algebra. The Hopf algebraic structure of $U_q[osp(2|2)^{(1)}]$ was then used to construct the exact S -matrix $S(\theta; \hat{\gamma})$ (up to CDD factors) for the fundamental vector representation. We argued that in the marginal limit this $U_q[osp(2|2)^{(1)}]$ S -matrix reduces to the exact physical S -matrix $S_Y(\theta)$ for the fundamental particles of the $osp(2|2)$ current-current model. We did not prove that the quantum group symmetry used to determine $S(\theta; \hat{\gamma})$ and $S_Y(\theta)$ is exact to all orders in g . Nevertheless, the fact that we were able to construct a S -matrix satisfying the scattering constraints and having a symmetry algebra agreeing with the Yangian is strong support for the validity of (4.130). As mentioned above, one can try to construct the Yangian charges and in turn determine $S_Y(\theta)$ from the Yangian symmetry. However, whereas

the procedure for computing S using quantum group symmetry is, in principle, well established, this is not the case for the Yangian symmetric situation. We hope to address this issue for the specific model (2.3) in the future. Another independent check of (4.130) will be to do a thermodynamic Bethe ansatz analysis [66, 67, 16, 18]. This is complicated by the fact that $S_Y(\theta)$ is not diagonal.

The $\hat{\beta}$ -dependent S -matrix $S(\theta; \hat{\gamma})$ is itself an interesting result. For $0 < 1/\hat{\gamma} < 1$, $S(\theta; \hat{\gamma})$ is the S -matrix for the supersymmetric Toda system (3.27), and reduces to the rational result $S_Y(\theta)$ at $1/\hat{\gamma} = 0$. Yet $S(\theta; \hat{\gamma})$ satisfies all the scattering constraints for any $1/\hat{\gamma} > 0$, which suggests that it is the fundamental “soliton” S -matrix for some other theory (recall that for $1/\hat{\gamma} > 1$ the Toda perturbation (3.29) becomes irrelevant). In the marginal limit this theory also flows to the $osp(2|2)$ current-current model. It is an open question to determine the theory, in the sense of an action, corresponding to (4.122) for all $\hat{\beta}$ (or $\hat{\gamma}$). This theory will be some perturbation of the $c = 0$ CFT (2.4). Since $S(\theta; \hat{\gamma})$ has a non-trivial pole structure, the complete S -matrix must include the scattering amplitudes for the neutral and charged bound states. These amplitudes can be found by applying the bootstrap principle.

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Appendix A

In this appendix we review the $osp(2|2)$ superalgebras and give the relationship between the $osp(2|2)^{(1)}$ generators and the currents (2.9), (2.10). Our discussion omits the Serre relations as they are not needed in this paper. For the same reason, we will also define the affine algebras without the derivation. Further details on general (affine) Lie superalgebras and quantum supergroups, including $osp(2|2)$, may be found in [51, 68–71]. We will follow most closely the notations and conventions of [51] and [69] (though unlike [51] and [69], we take a simple root system for $osp(2|2)$ that is purely fermionic).

(1) *The superalgebras $osp(2|2)$ and $osp(2|2)^{(1)}$*

The simple Lie superalgebra $osp(2|2)$ is a Z_2 -graded algebra with two simple roots, $\{\alpha_1, \alpha_2\}$, and Chevalley generators $\{e_i, f_i, h_i; i = 1, 2\}$ satisfying

$$[h_i, h_j] = 0 \tag{A.1a}$$

$$[h_i, e_j] = a_{ij}e_j, \quad [h_i, f_j] = -a_{ij}f_j \tag{A.1b}$$

$$[e_i, f_j] = \delta_{ij}h_i \tag{A.1c}$$

$$e_i^2 = f_i^2 = 0, \quad \text{if } a_{ii} = 0. \tag{A.1d}$$

Here (i) $[\cdot, \cdot]$ denotes the graded Lie bracket

$$[a, b] = ab - (-1)^{d(a)d(b)}ba, \tag{A.2}$$

where $d(x)$ is the parity of x : $d(x) = 0$ if x is even or bosonic and $d(x) = 1$ if x is odd or fermionic (all Cartan generators h_i are even); and (ii) a_{ij} is the symmetric generalized Cartan matrix defined as

$$a_{ij} = (\alpha_i, \alpha_j), \tag{A.3}$$

where (\cdot, \cdot) is a fixed invariant bilinear form on the root space. In contrast with (bosonic) Lie algebras, superalgebras allow several inequivalent simple root systems. A common choice is the distinguished root system, where all simple roots except one are taken to be bosonic. We will instead work with a purely fermionic $osp(2|2)$ simple root system with the parities

$$d_{1,2} \equiv d(e_{1,2}) = d(f_{1,2}) = 1, \quad \tilde{d}_{1,2} \equiv d(h_{1,2}) = 0. \tag{A.4}$$

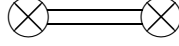
For the $osp(2|2)$ Cartan matrix we take

$$a_{ij} = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}. \quad (\text{A.5})$$

A specific realization of the root system is given by

$$\alpha_1 = (1, i), \quad \alpha_2 = (-1, i). \quad (\text{A.6})$$

The Dynkin diagram associated with (A.5) is



The untwisted affine Lie superalgebra $osp(2|2)^{(1)}$ contains an additional root $\alpha_0 = -\psi$, where ψ is the highest root of $osp(2|2)$. Explicitly we have

$$\alpha_0 = -(\alpha_1 + \alpha_2) = (0, -2i). \quad (\text{A.7})$$

This additional root is necessarily even, with $d_0 \equiv d(e_0) = d(f_0) = 0$. The affine generators $\{e_i, f_i, h_i; i = 0, 1, 2\}$ satisfy the same relations (A.1) as for $osp(2|2)$. (If we included the derivation, α_0 would be given by $\delta - \psi$, where δ is the minimal imaginary root of $osp(2|2)^{(1)}$. However, since δ satisfies $(\delta, \delta) = (\delta, \alpha_{1,2}) = 0$, the defining relations (A.1) are unchanged.) The affine Cartan matrix is

$$a_{ij} = \begin{pmatrix} -4 & 2 & 2 \\ 2 & 0 & -2 \\ 2 & -2 & 0 \end{pmatrix}. \quad (\text{A.8})$$

Note that in section 3 we find $h_0 = -(h_1 + h_2)$, thus there h_0 is not an independent generator. This implies that the central extension is zero. For a non-zero central extension all the Cartan generators $\{h_0, h_1, h_2\}$ satisfying (A.1) are independent.

(2) *The quantum superalgebras $U_q[osp(2|2)]$ and $U_q[osp(2|2)^{(1)}]$*

The quantum superalgebras (or supergroups) $U_q[osp(2|2)]$ and $U_q[osp(2|2)^{(1)}]$ are deformations of the universal enveloping algebras for $osp(2|2)$ and $osp(2|2)^{(1)}$. As such they are (unital) Z_2 -graded associative algebras generated by $\{e_i, f_i, h_i\}$, where $i = 1, 2$ for $U_q[osp(2|2)]$ and $i = 0, 1, 2$ for $U_q[osp(2|2)^{(1)}]$, modulo the relations

$$[h_i, h_j] = 0 \quad (\text{A.9a})$$

$$[h_i, e_j] = a_{ij}e_j, \quad [h_i, f_j] = -a_{ij}f_j \quad (\text{A.9b})$$

$$[e_i, f_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}} \quad (\text{A.9c})$$

$$e_i^2 = f_i^2 = 0, \quad \text{if } a_{ii} = 0. \quad (\text{A.9d})$$

Here q is an arbitrary non-zero complex number. In the limit $q \rightarrow 1$, (A.9) reduces to (A.1). The expressions $q^{\pm h_i}$ are understood as infinite power series in h_i . We can alternatively write (A.9a) and (A.9b) as

$$q^h q^{h'} = q^{h'} q^h, \quad h, h' \in \{\pm h_i\} \quad (\text{A.10a})$$

$$q^{h_i} q^{-h_i} = q^{-h_i} q^{h_i} = 1 \quad (\text{A.10b})$$

$$q^{h_i} e_j q^{-h_i} = q^{a_{ij}} e_j, \quad q^{h_i} f_j q^{-h_i} = q^{-a_{ij}} f_j. \quad (\text{A.10c})$$

These quantum algebras can be endowed with a Hopf algebraic structure. The comultiplication Δ is defined as

$$\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i \quad (\text{A.11a})$$

$$\Delta(e_i) = e_i \otimes q^{-h_i/2} + q^{h_i/2} \otimes e_i \quad (\text{A.11b})$$

$$\Delta(f_i) = f_i \otimes q^{-h_i/2} + q^{h_i/2} \otimes f_i, \quad (\text{A.11c})$$

or equivalently in place of (A.11a)

$$\Delta(q^{\pm h_i}) = q^{\pm h_i} \otimes q^{\pm h_i}. \quad (\text{A.12})$$

The antipode \mathcal{S} and counit ε are

$$\begin{aligned} \mathcal{S}(h_i) &= -h_i \\ \mathcal{S}(e_i) &= -q^{-(\alpha_i, \alpha_i)/2} e_i \\ \mathcal{S}(f_i) &= -q^{(\alpha_i, \alpha_i)/2} f_i \end{aligned} \quad (\text{A.13})$$

$$\varepsilon(h_i) = \varepsilon(e_i) = \varepsilon(f_i) = 0, \quad \varepsilon(1) = 1. \quad (\text{A.14})$$

A thorough discussion of the representation theory for the $osp(2|2)$ algebras can be found in [43–47, 49].

(3) *The $osp(2|2)^{(1)}$ currents*

The super-currents (2.9) have Laurent expansions of the form (concentrating only on the holomorphic sector)

$$\mathcal{O}(z) = \sum_n z^{-n-1} \mathcal{O}_n. \quad (\text{A.15})$$

The algebra satisfied by the modes $\{\mathcal{O}_n\}$ can be obtained using the OPE's (2.11) and the formula

$$[A, B] = \oint_0 d\omega \oint_\omega dz a(z)b(\omega), \quad (\text{A.16})$$

for

$$A = \oint_0 dz a(z), \quad B = \oint_0 dz b(z). \quad (\text{A.17})$$

For the modes \mathcal{O}_m and \mathcal{O}'_n , the (graded) commutator is

$$[\mathcal{O}_m, \mathcal{O}'_n] = \oint_0 d\omega \omega^n \oint_\omega dz z^m \mathcal{O}(z) \mathcal{O}'(\omega). \quad (\text{A.18})$$

To prove the equivalence between the current algebra and the level one $osp(2|2)^{(1)}$ algebra, we need to show that (A.18) agrees with (A.1). This amounts to correctly identifying the modes with the generators $\{e_i, f_i, h_i; i = 0, 1, 2\}$. We find the following relations (here $(\mathcal{O})_m \equiv \mathcal{O}_m$)

$$\begin{aligned} e_0 &= \frac{1}{\sqrt{2}} \hat{K}_{-1}, & f_0 &= \frac{1}{\sqrt{2}} K_{+1}, & h_0 &= 1 + 2H_0 \\ e_1 &= i(G_-)_0, & f_1 &= i(\hat{G}_+)_0, & h_1 &= -(H_0 + J_0) \\ e_2 &= i(G_+)_0, & f_1 &= i(\hat{G}_-)_0, & h_2 &= -(H_0 - J_0). \end{aligned} \quad (\text{A.19})$$

Note that $h_0 \neq -(h_1 + h_2)$, but rather $h_0 = k - (h_1 + h_2)$ where $k = 1$ is the level. This differs from the quantum group charges (3.56) where the level, or central extension, is zero and $T^0 = -(T^1 + T^2)$. One can check that (A.19) is consistent with (2.11) and (A.1).

In the Cartan-Weyl basis, the conserved currents for the conformal field theory (2.4) are

$$\begin{aligned} \mathcal{J}(z) &= e_0 \frac{1}{\sqrt{2}} \hat{K} + e_1(iG_-) + e_2(iG_+) + f_0 \frac{1}{\sqrt{2}} K + f_1(i\hat{G}_-) + f_2(i\hat{G}_+) + \frac{1}{2}(h_2 - h_1)J - \frac{1}{2}(h_2 + h_1)H \\ \overline{\mathcal{J}}(\bar{z}) &= e_0 \frac{1}{\sqrt{2}} \overline{\hat{K}} + e_1(-i\overline{G}_-) + e_2(-i\overline{G}_+) + f_0 \frac{1}{\sqrt{2}} \overline{K} + f_1(-i\overline{\hat{G}}_+) + f_2(-i\overline{\hat{G}}_-) + \frac{1}{2}(h_2 - h_1)\overline{J} - \frac{1}{2}(h_2 + h_1)\overline{H}. \end{aligned} \quad (\text{A.20})$$

A current-current interaction $\Phi_{\text{int}}^{\text{cc}}$ is given by

$$\Phi_{\text{int}}^{\text{cc}} \propto \text{str}(\overline{\mathcal{J}}\mathcal{J}), \quad (\text{A.21})$$

where the constant of proportionality depends on the particular representation. Evaluating (A.21) using the representation matrices (3.59) and (3.61) we get

$$\text{str}(\overline{\mathcal{J}}\mathcal{J}) = -2 \left[\overline{J}J - \overline{H}H + \frac{1}{2}(\overline{K}\hat{K} + \overline{\hat{K}}K) + \overline{G}_-\hat{G}_+ - \overline{\hat{G}}_-G_+ + \overline{G}_+\hat{G}_- - \overline{\hat{G}}_+G_- \right]. \quad (\text{A.22})$$

This is identical to (2.12) hence confirming that Φ_V is indeed a current-current perturbation.

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